2012-2013 – Master 2 – Macro I

Lecture 3 : The Ramsey Growth Model

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Version 1.1
07/10/2012
Changes from version 1.0 are in red
These are the slides I am using in class. They are not self-contained, do not always constitute original material and do contain some “cut and paste” pieces from various sources that I am not always explicitly referring to (not on purpose but because it takes time). Therefore, they are not intended to be used outside of the course or to be distributed. Thank you for signalling me typos or mistakes at franck.portier@TSE-fr.eu.
The Ramsey model is a model of optimal growth for a closed, neo-classical economy populated by a representative consumer. While the Solow model assumes a fixed savings rate, in the Ramsey model the savings behaviour comes from the intertemporal optimization by consumers.

Strictly speaking, the Ramsey model is a normative model: it tells us what the optimal growth path is.

However, it can be interpreted as a positive model of how the economy actually evolves.

The reason is that if markets are complete and competition is perfect, any Walrasian equilibrium must be Pareto-optimal. But if there is a single representative agent, there is a unique Pareto-optimum: the one that maximizes the utility of the representative agent. Therefore, the optimal path of the Ramsey model is also an equilibrium; one under those assumptions.
1. The Model

Fundamentals

- There is a representative consumer whose utility is

\[ V = \int_0^{+\infty} e^{-\rho t} \frac{C_t^\alpha - 1}{\alpha} dt. \]

\( \rho \) is the rate of time preference and \( \alpha \in (-\infty, 1] \) the exponent parameter in the utility function. The *intertemporal elasticity of substitution* is then equal to

\[ \sigma = 1/(1 - \alpha). \]

- The production function is

\[ Y_t = F(K_t, A_t L_t). \]  

\( K_t \) is the capital stock, \( L_t \) is total employment, and \( A_t \) an exogenous trend of labor-augmenting technical progress.
1. The Model

Fundamentals (continued)

- We assume that $F$ is concave with constant returns to scale and satisfies the Inada conditions:

\[
\begin{align*}
\frac{\partial F}{\partial L}(K,0) &= \frac{\partial F}{\partial K}(0,L) = +\infty \\
\frac{\partial F}{\partial L}(K,\infty) &= \frac{\partial F}{\partial K}(\infty,L) = 0
\end{align*}
\]

Also: $F(K,0) = F(0,L) = 0$.

- We assume that $\dot{A}/A = g$, so that

\[ A_t = A_0 e^{gt}. \]

- Finally the equation for capital accumulation is

\[ \dot{K}_t = Y_t - \delta K_t - C_t, \quad \text{(KA)} \]

where $\delta =$ depreciation of capital.

- In what follows we will assume that employment is constant and equal to $\bar{L}$. 

Exercise: Let $\bar{F}(K, AL) = F(K, AL) - \delta K$ be the net GDP production function. Does it satisfy the Inada conditions? How would the model be rewritten if net GDP $\bar{Y} = Y - \delta K$ were used instead of gross GDP?
2. Solving the model

To solve the model we apply the optimal control technique. The Hamiltonian is

\[ H_t = e^{-\rho t} \frac{C_t^\alpha - 1}{\alpha} + \lambda_t [F(K_t, A_t L_t) - \delta K_t - C_t]. \]

The optimality conditions are

\[ \frac{\partial H}{\partial C} = 0 \iff e^{-\rho t} C_t^{\alpha-1} = \lambda_t. \]

This means that the marginal utility of consumption is equal to the marginal benefit of its alternative use, i.e. the value of one unit of capital.

\[ \frac{\partial H}{\partial K} = -\dot{\lambda} \iff \lambda \left( \frac{\partial F}{\partial K} - \delta \right) = -\dot{\lambda}. \]
2. Solving the model

Co-state variable

- Since $\lambda$ is also equal to the marginal value of consumption, this means that

$$
\lambda_t = \int_t^{+\infty} e^{-\rho v} C_v^{\alpha-1} \left( \frac{\partial F}{\partial K} - \delta \right) dv.
$$

- Therefore, $\lambda_t$ is the net value of the future flow of marginal products of capital, net of depreciation, expressed in utility terms. This is indeed what I would get if God gave me an extra unit of capital for free at date $t$.

- We can introduce $\mu_t$, the contemporaneous co-state variable, such that $\lambda_t = \mu_t e^{-\rho t}$. We get

$$
C_t^{\alpha-1} = \mu_t \quad (1)
$$

and

$$
\mu_t \left( \frac{\partial F}{\partial K}(K_t, A_t L_t) - \delta \right) = -\dot{\mu}_t + \rho \mu_t. \quad (2)
$$
2. Solving the model
A digression on asset values

Consider an asset which pays a dividend $d_t$ and whose value at $t$ is $V_t$. Assume that I can get a rate of return elsewhere equal to $r$. This implies that the flow of dividends will be discounted at rate $r$.

To see this, note that if I sell my asset now and invest the proceeds at rate $r$, I will get $(1 + r)V_t$ tomorrow. If I hold it, I will get $d_t + V_{t+1}$. Therefore the arbitrage condition is

$$V_t = \frac{d_t + V_{t+1}}{1 + r}. \quad (3)$$

Equivalently we can write:

$$rV_t = d_t + (V_{t+1} - V_t). \quad (4)$$
2. Solving the model
A digression on asset values (continued)

- That is:

  \[
  \text{Rate of Return} \times \text{Value of the asset} = \text{Dividends} + \text{Capital Gains}.
  \]

- This means that the rate of return on the asset must be equal to the rate of return on the alternative investment, otherwise there would be arbitrage possibilities.

- Note that for risk neutral agents, the formula extends to risky investments:

  \[
  rV_t = d_t + E_t(V_{t+1} - V_t).
  \]

- That is:

  \[
  \text{Rate of Return} \times \text{Value of the asset} = \text{Dividends} + \text{Expected Capital Gains}
  \]
2. Solving the model

A digression on asset values (continued)

▶ An equation like (3) can be solved by *forward integration*

\[
V_t = \frac{1}{1 + r} \sum_{i=t}^{T} \frac{1}{(1 + r)^{i-t}} d_i + \frac{V_{T+1}}{(1 + r)^{T+1-t}}. \tag{5}
\]

▶ Assume that the following sequence:

\[
F_t = \frac{1}{1 + r} \sum_{i=t}^{\infty} \frac{1}{(1 + r)^{i-t}} d_i \tag{6}
\]

exists. Then if \( r > 0 \) \( F_t \) is the only non explosive solution to (5). So, if we eliminate explosive solutions, then we have a unique one which is called the *fundamental* value of the asset.

▶ Furthermore, let \( \tilde{V}_t \) be any other solution. Let \( B_t = \tilde{V}_t - F_t \). It must be that

\[
B_{t+1} = (1 + r)B_t.
\]

▶ Therefore, any solution is the sum of the "fundamental" and a "bubble" which explodes at rate \( r \).
2. Solving the model

A digression on asset values (continued)

- **Remark**: In fact the bubble can be stochastic and only need to explode at rate $r$ in expectations. That is, up to a multiplicative factor $(1 + r)$ the bubble can be any martingale.

- **Remark**: Conversely, any present discounted value of an income flow can be rewritten in a recursive fashion. For example if

$$W_t = \sum_{i=t}^{+\infty} \frac{1}{(1 + r)^{i-t}} y_i,$$  \hspace{1cm} (7)

one can always express $W_t$ as a function of the income stream today and tomorrow’s value $W_{t+1}$:

$$W_t = y_t + \frac{1}{1 + r} W_{t+1}.$$  \hspace{1cm} (8)

Then this can be again interpreted in a dividend plus capital gains fashion:

$$rW_t = (1 + r)y_t + W_{t+1} - W_t.$$  \hspace{1cm} (9)
2. Solving the model
A digression on asset values (continued)

▶ Exercise: Why are (7), (8), and (9) not quite similar to (6), (3) and (4)? What does it have to do with the timing of dividend payments?
2. Solving the model

A digression on asset values (continued)

- The preceding derivations can be made in continuous time.
- Suppose an asset yields a flow of dividends per unit of time equal to \( x(t) \).
- Suppose I can get a return \( r \) per unit of time elsewhere.
- Let \( V(t) \) be the value of the asset at \( t \).
- Suppose I sell it and invest in the market over a small time interval \( dt \).
- Then at \( t + dt \) I will have \( V(t)(1 + r.dt) \).
- If instead I hold the asset, then at \( t + dt \) I will have \( x(t).dt + V(t + dt) \).
- Hence
  \[
  x(t).dt + V(t + dt) = V(t)(1 + r.dt),
  \]
- that is, neglecting second order terms:
  \[
  rV(t) = x(t) + \dot{V}(t)
  \]
2. Solving the model
A digression on asset values (continued)

\[ rV(t) = x(t) + \dot{V}(t) \]

- The LHS is the rate of return (per unit of time) times the value of the asset.
- The RHS is the dividend per unit of time, plus the expected capital gain per unit of time.
- The equation is the same as (4), except that the arbitrage takes place over a small time interval and that all is expressed per unit of time.
- In continuous time the fundamental is

\[ F_t = \int_t^{+\infty} e^{-r(v-t)} x(v) dv \]

- And a bubble is such that

\[ \dot{B} = rB \quad \text{that is} \quad B_t = B_0 e^{rt}. \]
2. Solving the model
A digression on asset values (continued)

▶ Exercise: How are the above derivations altered when the instantaneous discount rate is time-dependent?

▶ Recall

$$rV_t = d_t + (V_{t+1} - V_t). \quad (4)$$

$$rW_t = (1 + r)y_t + W_{t+1} - W_t. \quad (9)$$

▶ Exercise: Derive the continuous-time equivalent of (9) for

$$W_t = \int_t^{+\infty} e^{-r(v-t)}x(v)dv.$$ Why is it that in continuous time there is no longer a discrepancy like the one between (9) and (4)?
2. Solving the model
A digression on asset values (continued)

Throughout the course we will see that any dynamic equation which drives the evolution of a "price" variable can be interpreted as an arbitrage equation on an asset.

This is true also for the marginal social values (=co-state variables) in a dynamic optimization problem. For example, take (2).

\[ \mu_t \left( \frac{\partial F}{\partial K}(K_t, A_t L_t) - \delta \right) = -\dot{\mu}_t + \rho \mu_t. \]  \hspace{1cm} (2)

We know that \( \mu_t \) is the marginal value of an extra unit of capital at date \( t \), expressed in utility terms and discounted at the current date \( t \).

Also, the required rate of return in utility terms is \( \rho \) : To be willing to sacrifice one unit of utility at \( t \) I need to get \( e^{\rho(t' - t)} \) units of utility at a future date \( t' \).
2. Solving the model

A digression on asset values (continued)

\[
\mu_t \left( \frac{\partial F}{\partial K} (K_t, A_t L_t) - \delta \right) = -\dot{\mu}_t + \rho \mu_t.
\] (2)

So equation (2) can be rewritten

\[
\rho \mu_t = x_t + \dot{\mu}_t,
\]

where

\[
x_t = \mu_t \left( \frac{\partial F}{\partial K} (K_t, A_t L_t) - \delta \right) = C_v^{\alpha-1} \left( \frac{\partial F}{\partial K} (K_t, A_t L_t) - \delta \right)
\]

- $x_t$ is the marginal utility of consumption times the net marginal product of capital
- $x_t$ is the dividend in hedonic terms of one extra unit of capital, and $\dot{\mu}_t$ is the capital gain.
2. Solving the model

The transversality condition

- The transversality condition is

\[
\lim_{t \to +\infty} \lambda_t K_t = \lim_{t \to +\infty} e^{-\rho t} \mu_t K_t = 0.
\]
2. Solving the model

The Euler condition

- We can eliminate \( \mu_t \) between (2) and (1) to get a dynamic relationship between consumption and capital. We have that

\[
\frac{\dot{\mu}}{\mu} = -\frac{1}{\sigma} \frac{\dot{C}}{C}
\]

and

\[
\frac{\dot{\mu}}{\mu} = \rho + \delta - \frac{\partial F}{\partial K}(K_t, A_t L_t).
\]

- Therefore

\[
\frac{\dot{C}}{C} = \sigma \left( \frac{\partial F}{\partial K}(K_t, A_t L_t) - \delta - \rho \right). \quad \text{(ec)}
\]

or

Growth rate of consumption

\[= \text{Intertemporal elasticity of substitution} \times \]

(\text{net marginal product of capital - rate of time preference}).

- This is the standard condition that comes out of partial-equilibrium dynamic consumption problems.
2. Solving the model

Renormalization

- We know from previous lecture that there exists a balanced growth path with growth rate $g$.
- A standard trick is to renormalize the variables by dividing them by the growth trend of productivity $A$.
- In a BGP the renormalized variables are therefore constant.
- For any variable $X_t$ in the model let us define $x_t = X_t / A_t$.
- Other renormalizations are possible provided they all deliver constant renormalized variables in a BGP.
- **Exercise**: Based on what you know about a BGP, propose 3 alternative renormalizations that have this property.
2. Solving the model

Renormalization

▶ Because of constant returns, we can rewrite $F$ as

$$F(K_t, A_t L_t) = A_t L_t f\left(\frac{K_t}{A_t L_t}\right).$$

▶ From the renormalization we get the following:

$$y_t = \bar{L} f\left(\frac{k_t}{L}\right),$$  \hspace{1cm} (pf)  

$$\dot{k}_t = y_t - (\delta + g)k_t - c_t.$$  \hspace{1cm} (ka)  

▶ Note also that

$$\frac{\partial F}{\partial K}(K_t, A_t L_t) = f'(\frac{k_t}{L})$$

▶ This allows us to derive the renormalized Euler condition:

$$\frac{\dot{c}_t}{c_t} = \sigma \left(f'(\frac{k_t}{L}) - \delta - \rho\right) - g.$$ \hspace{1cm} (ec)
3. The balanced growth path

Long run capital stock

- In the BGP, renormalized variables are constant. Therefore, the BGP is characterized by the following condition:

\[ f'(\frac{k_t}{L}) - \delta = \rho + \frac{g}{\sigma}. \] (MGR)

This pins down the capital stock in the long run.

- In words:

  Net marginal product of capital
  
  \[ = \text{rate of time preference} + \text{growth rate/intertemporal elasticity of substitution}. \]
3. The balanced growth path

Long run capital stock (continued)

\[ f'(\frac{k_t}{L}) - \delta = \rho + \frac{g}{\sigma}. \]  \hspace{1cm} (MGR)

- Interpretation:
  - The greater \( \rho \), the more people, prefer to consume now rather than later. As a result they save less and accumulate less capital.
  - The greater \( g \), the more they must consume in the future relative to now, and therefore the greater the return on capital necessary to induce them to do so. Because of decreasing returns to capital this means a lower level of capital relative to the growth trend. (A difficulty here is that an increase in \( g \) affects our normalization).

- Exercise: Can there be a balanced growth path if utility is not a power function of consumption?
3. The balanced growth path

The Modified Golden Rule

- The problem with that kind of interpretation is that it tells us what must hold in equilibrium but it does not tell us by what economic mechanism that is achieved.
- Assume growth accelerates. Holding the return to capital constant, people would want to borrow against the higher future income to consume more today. But in equilibrium this must come at the expense of capital accumulation: we accumulate less capital, this increases the return to capital and makes us less averse to postponing consumption.
- In equilibrium we therefore has less capital and higher consumption growth than before.
- The greater $\sigma$, the less sensitive to growth is the marginal product of capital. This is because if $\sigma$ is large, only a small increase in that marginal product is required to induce consumers to postpone consumption by enough in order to keep up with the growth trend.
- (MGR) : modified golden rule or ”Keynes-Ramsey” condition.
A key result is that the economy converges to the balanced growth path.

This is illustrated by a phase diagram (fig. 1).

The $\dot{c} = 0$ schedule corresponds to (ec), while the $\dot{k} = 0$ schedule comes from (ka), where $y$ is replaced by $\tilde{L}f(\frac{k_t}{L})$.

The arrows define the law of motion of the economy in the $(k, c)$ space and come from those equations.
4. Convergence to the balanced growth path

Figure 1: The Saddle-Path
4. Convergence to the balanced growth path

Uniqueness

- At any date $t$ the capital stock is determined from the past.
- In other words it cannot jump and must evolve smoothly according to the differential equation (ka).
- On the other hand, at a given initial date consumption may be anything.
- But thereafter it must also evolve smoothly according to (ec). Thus consumption cannot jump in a perfectly anticipated way because this would violate (ec). But its initial value is not pinned down by the past.
- In principle there is an infinity of possible trajectories, each corresponding to a choice for the initial consumption $c$. 
4. Convergence to the balanced growth path
Uniqueness (continued)

However:

- If initial consumption is too high, then the subsequent trajectory implies that capital is exhausted in finite time. Then consumption hits its zero boundary and must remain equal to zero thereafter. Consequently, it is not differentiable at the date when capital becomes equal to zero which violates \( \text{(ec)} \). These paths cannot be optimal.

- If initial consumption is too low, the subsequent trajectory involves a normalized consumption which goes to zero. Picking a slightly higher consumption level would yield higher consumption at each date, which clearly dominates the initial trajectory. Therefore the initial trajectory cannot be optimal. Technically, the condition which is violated is the transversality condition.
4. Convergence to the balanced growth path

Proof of uniqueness

- Asymptotically we have, along such paths, that $c = 0$.
- The asymptotic value of $k$ is $\bar{k}$ such that $\bar{L}f(\frac{\bar{k}}{L}) = (\delta + g)\bar{k}$.
- By concavity, $\bar{L}f(\frac{k}{L}) > kf'(\frac{k}{L})$.
- Thus $f'(\frac{\bar{k}}{L}) < \delta + g$. Let $z = \delta + g - f'(\frac{\bar{k}}{L}) > 0$.
- Substituting into (2) we then get that

$$\dot{\mu}_t = (z + \rho - g)\mu_t.$$ 

Hence:

$$\mu_t = \mu_{t_0} \exp((z + \rho - g)(t - t_0)).$$

- Finally,

$$e^{-\rho t}\mu_t K_t = e^{-\rho t} \mu_{t_0} \exp((z + \rho - g)(t - t_0))\bar{k}A_t$$

$$= e^{-\rho t} \mu_{t_0} \exp((z + \rho - g)(t - t_0))\bar{k}A_0 e^{gt}$$

$$= M.e^{zt} \to \infty,$$

where $M$ is a constant.
4. Convergence to the balanced growth path

Interpretation of the proof

➤ At $\bar{k}$ the net marginal product of capital is lower than $g$.
➤ This means that by reducing $\bar{k}$ I would lose less than the fraction of GDP I need just to maintain $\bar{k}$ in line with the growth trend. This would free a surplus that I could consume.
➤ So I will only keep this capital if its marginal value is expected to appreciate by enough to offset that negative effect.
➤ But then the marginal value of capital must explode at such a rate that the transversality condition is violated, (meaning that I am keeping too much capital in the long run)
➤ Thus we see that the only acceptable trajectory is the one which converges to the steady state in the $(k, c)$ plane, i.e. the BGP of the economy. This allows to pin down the initial value of consumption.
➤ Consequently, not only does the BGP exist, but the economy converges to this BGP for any initial value of $K$.
➤ Note: The trajectory which converges to the BGP is called the saddle path.
5. Location of the Modified Golden Rule

On the left of Golden Rule

- One can show that the modified golden rule of the capital stock is always to the left of, i.e. smaller than, the maximum point of the $\dot{k} = 0$ schedule.
- Note that this maximum is the one which maximizes
  \[ c = y - (\delta + g)k = \bar{L}f\left(\frac{k_t}{\bar{L}}\right) - (\delta + g)k. \]
- The FOC is $f'(k/\bar{L}) = \delta + g$. Let $k^*$ be the MGR capital level and $\bar{k}$ be the one corresponding to the top of $\dot{k} = 0$.
- Then $k^* < \bar{k}$ if and only if:
  \[ \rho + \frac{g}{\sigma} > g, \]
- or equivalently, since $\sigma = 1/(1 - \alpha)$,
  \[ \alpha g < \rho. \]
5. Location of the Modified Golden Rule

Interpretation

- Consider the utility of the consumer in a BGP. Up to a constant, it is equal to

\[
\int_{0}^{+\infty} \frac{(c_{0}e^{gt})^{\alpha}}{\alpha} e^{-\rho t} dt = \int_{0}^{+\infty} \frac{c_{0}^{\alpha}}{\alpha} e^{(g\alpha-\rho)t} dt.
\]

- Clearly, this exists iff (10) holds.
- The meaning of this is that if (10) is violated, the growth potential of the economy is such that an infinite amount of utility can be obtained because the contribution of future consumption to utility grows faster than the rate of time preference.
6. Comparative dynamics
An increase in the rate of time preference.

- People are more impatient and start consuming more as the return to capital gets lower than their rate of time preference.
- As capital gets depleted, the economy becomes less productive and the MPK goes up.
- In equilibrium it has increased by the same amount as the rate of time preference and there is no longer an incentive to deplete it.
- The capital stock is lower and so must be consumption (otherwise we would be on a capital exhaustion trajectory).
Comparative dynamics

Figure 2: An increase in the rate of time preference (continued)

Figure 2: Impact of a rise in $\rho$
6. Comparative dynamics

An additive productivity shock ("manna from heaven")

- Consumption jumps immediately by an amount equal to the manna.
- The manna is entirely consumed and the capital stock is unchanged.
- Starting from the initial capital stock, at the given rate of return on capital consumers see an increase in their income by the same amount in all current and future periods.
- Thus their current income and their permanent income increase by the same amount.
- At this rate of return they thus want to increase their consumption permanently by that amount.
- As this leaves the capital stock unchanged, the rate of return is unchanged and we indeed have an equilibrium outcome.
Comparative dynamics

Figure 3: An additive productivity shock ("manna from heaven") (continued)
6. Comparative dynamics
A multiplicative productivity shock

- Both schedules shift out.
- In the long term there is more capital and more consumption. This is because capital is more productive, so it is worth accumulating more of it and use the proceeds to consume more in the long-run.
- In the short run, the effect is ambiguous: there is a substitution effect which tells us that it is worth consuming less today and more tomorrow because accumulating more capital is now a better idea.
- There is an income effect which says that we will have more income throughout the whole trajectory and we may want to consume a little bit more right now.
6. Comparative dynamics

Figure 4: A multiplicative productivity shock (continued)
6. Comparative dynamics

An anticipated manna from heaven

- At date $t = 0$ it is known that at some future date $T$ the economy will undergo a permanent shock in one of the parameters.
- The key points here are that
  - starting from $T$ the economy must be on the new saddle path
  - consumption cannot jump in perfectly anticipated fashion because this would violate the optimality condition (ec)
- **Remark**: By contrast, consumption may jump if there are news, because when there are news people *re-optimize*. This is why consumption jumps upon the shock in all the preceding examples, and will jump at $t = 0$ here).
- We see that between the announcement and the realization of the shock consumption goes up and the capital stock is being gradually depleted.
- At the date of the shock we are exactly on the new saddle path, so that consumption continues to grow but now we use the manna to reaccumulate capital.
6. Comparative dynamics
An anticipated manna from heaven (continued)

- Why is this optimal? If we were waiting for the shock to materialize in order to consume more, consumption would not be smooth.
- Relative to this, we can increase our utility by consuming more now and less when the shock occurs.
- As a consequence, we have less capital at the time of the shock, but this raises its return above the rate of time preference (adjusted for growth), so it makes sense to gradually rebuild it.
6. Comparative dynamics

Figure 5: An anticipated manna from heaven (continued)

Figure 5: Anticipated manna from heaven