

2018-2019 – Econ 0107 – Macroeconomics I

Lecture 3 : Complete Markets

(Chapter 8 in LJUNQVIST & SARGENT)

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Changes from version 1.0 are in red

1. Time 0 versus Sequential Trading

- ▶ This chapter describes competitive equilibria for a pure exchange infinite horizon economy with stochastic endowments.
- ▶ This economy is useful for studying risk sharing, asset pricing, and consumption.
- ▶ Two market structures:
 - × ARROW-DEBREU structure with complete markets in dated contingent claims all traded at time 0,
 - × sequential-trading structure with complete one-period ARROW securities
- ▶ These two imply different assets and timings of trades, but have identical consumption allocation.

2. Endowments and Preferences

We start by characterizing:

1. The physical (and stochastic) properties of the resources available to the agents.
2. Agents preferences.

2. Endowments and Preferences

Histories

- ▶ We consider a stochastic event or state variable $s_t \in S$.
- ▶ Let's define the history of events up to time t as:

$$s^t = \{s_0, s_1, \dots, s_t\}$$

- ▶ The unconditional probability of observing **state s_t in** history s^t is

$$\pi_t(s^t)$$

- ▶ Conditional probabilities, for history s^t conditional on history s^τ , $\tau < t$, are given by:

$$\pi_t(s^t | s^\tau)$$

- ▶ Note: we will not immediately assume a Markov structure.
 - × While useful for some results, it is not necessary for others.

Endowments

- ▶ We assume the economy is populated by I agents, denoted by index $i = 1, 2, \dots, I$.
- ▶ Each agent receives an endowment which is a function of the random history s^t :

$$y_t^i(s^t).$$

- ▶ Each household will be associated with a consumption plan

$$c_i = \{c_t^i(s^t)\}_{t=0}^{\infty}.$$

2. Endowments and Preferences

Preferences

- ▶ Consumption plans will be ordered using the following function:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t) = E_0 \sum_{t=0}^{\infty} \beta^t u[c_t^i(s^t)].$$

- ▶ Regularity conditions on $u[\cdot]$.
 - × Increasing, concave and smooth.
 - × Inada condition.

$$\lim_{c \downarrow 0} u'(c) = +\infty$$

- ▶ Consumers share probabilities (views of the world)

2. Endowments and Preferences

Allocations

- ▶ All histories are fully observable, verifiable and contractable upon.

Definition 1

An **allocation** of resources is a collection of consumption plans $C = \{c_i\}_{i=1}^I$.

Definition 2

An allocation of resources is **feasible** if:

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \quad \forall t \text{ and } \forall s^t.$$

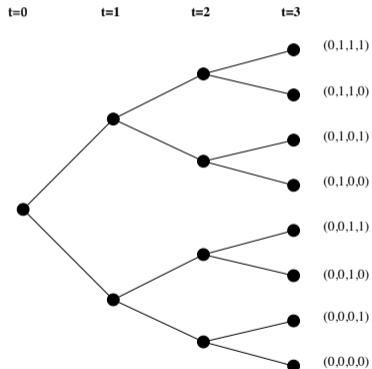
- ▶ Notice: we are not considering storage.

3. Alternative Trading Arrangements

- ▶ We will consider two different trading arrangements.
 - ▶ We will then show their equivalence.
1. We will first consider a situation where all trades happen at time 0.
 - × Agents will trade state and history contingent claims.
 - × Once the trade is done, markets close for subsequent periods.
 - × Contracts are enforced in subsequent periods and commitments honored.
 2. We will then consider a sequential trading framework.
 - × In each period agents will trade one-period contingent claims.

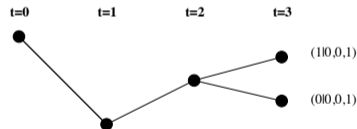
3. Alternative Trading Arrangements

Figure 1: Time 0 ARROW-DEBREU trades: all possible histories are considered at time 0.



The ARROW-DEBREU commodity space for a two-state Markov chain. At time 0, there are trades in time $t = 3$ goods for each of the eight nodes that signify histories that can possibly be reached starting from the node at time 0.

Figure 2: Sequential trades in 1-period ARROW securities



The commodity space with ARROW securities. At date $t = 2$, there are trades in time 3 goods for only those time $t = 3$ nodes that can be reached from the realized time $t=2$ history $(0,0,1)$.

4. PARETO Problem

Definition 3

An allocation is said to be PARETO-optimal if any re-allocation that makes one household strictly better off also makes one or more other households worse off.

Definition 4

An allocation is said to be efficient if it is PARETO-optimal.

- ▶ We can construct efficient allocations from a PARETO problem:
- ▶ A social planner maximizes the weighted average of individual utilities using some arbitrary non-negative PARETO weights $\lambda_i, i = 1, \dots, I$.

$$W = \sum_{i=1}^I \lambda_i U(c^i) \quad \text{subject to}$$
$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \quad \forall t \text{ and } \forall s^t.$$

4. PARETO Problem

- ▶ Consider the Lagrangian for this problem:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^I \lambda_i \beta^t u(c_t^i(s^t)) \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^I [y_t^i(s^t) - c_t^i(s^t)] \right\}$$

where $\theta_t(s^t)$ are (non-negative) Lagrange multipliers associated to the feasibility constraints relevant in each state and time.

- ▶ First order condition w.r.t. $c_t^i(s^t)$:

$$\beta^t u'(c_t^i(s^t)) \pi_t(s^t) = \lambda_i^{-1} \theta_t(s^t) \quad \forall i, t, s^t.$$

- ▶ Considering the ratio of this condition to that for consumer 1:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i}$$

4. PARETO Problem

- ▶ Solving for $c_t^i(s^t)$:

$$c_t^i(s^t) = u'^{-1}\left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))\right).$$

- ▶ Substituting in the feasibility constraint:

$$\sum_i u'^{-1}\left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))\right) = \sum_i y_t^i(s^t).$$

Proposition 1

An efficient allocation is a function of the realized aggregate endowment and depends neither on the specific history leading up to that outcome nor on the realizations of individual endowments.

$$c_t^i(s^t) = c_\tau^i(\bar{s}^\tau) \quad \forall s^t \quad \text{and} \quad \bar{s}^\tau \quad \text{such that} \quad \sum_j y_t^j(s^t) = \sum_j y_\tau^j(\bar{s}^\tau).$$

4. PARETO Problem

Notice:

- ▶ PARETO weights can be normalized (only ratios matter) \rightsquigarrow we can impose $\sum_i \lambda_i = 1$.
- ▶ PARETO weights are time invariant.
- ▶ Relative marginal utilities depend on PARETO weights only, so that they are time invariant.
- ▶ Consumer's i relative share of the aggregate endowment varies with his PARETO weight λ_i , and is invariant.
- ▶ Constancy of the PARETO weights indicates that there is no problem of enforcement or information-related incentive problems.
- ▶ So far we have described the allocations, not the specific trades made to achieve those allocations.

5. Time-0 trading: ARROW-DEBREU securities.

- ▶ Here we describe a particular market organisation that reaches a PARETO-efficient allocation.
- ▶ At time zero consumers trade entitlements to state-contingent consumption streams.
- ▶ The price of a security that delivers one unit of consumption at time t if the history up to that point has been s^t is $q_t^0(s^t)$.
- ▶ In other words, $q_t^0(s^t)$ is the price at time 0 of time t consumption contingent on history s^t , in terms of an abstract unit of account.
- ▶ The budget constraint for a consumer entering the time 0 market will then be:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t).$$

- ▶ Consumer i will maximize utility given this budget constraint.
- ▶ Notice: there is a single budget constraint because all trades occur at time 0.

5. Time-0 trading: ARROW-DEBREU securities.

- ▶ For a generic household i

$$\max U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) \quad (\mu_i)$$

- ▶ First order condition will be:

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \mu_i q_t^0(s^t).$$

μ_i is the Lagrange multiplier associated with budget constraint.

- ▶ Given intertemporal separability of preferences,

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \beta^t u'[c_t^i(s^t)] \pi_t(s^t) = \mu_i q_t^0(s^t).$$

5. Time-0 trading: ARROW-DEBREU securities.

Definition 5

A **price system** is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$.

Definition 6

An **allocation** is a list of sequence of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$.

Definition 7

A **competitive equilibrium** is a feasible allocation and a price system such that, given the price system, the allocation solves each household's problem.

5. Time-0 trading: ARROW-DEBREU securities.

- Notice that:

$$\beta^t u'[c_t^i(s^t)] \pi_t(s^t) = \mu_i q_t^0(s^t)$$

implies:

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^j(s^t)]} = \frac{\mu_i}{\mu_j}, \quad \forall i, j$$

- An equilibrium, therefore, solves:
1. The aggregate feasibility constraint.
 2. The individual budget constraint for all individuals.
 3. The condition on the ratios of marginal utilities.

5. Time-0 trading: ARROW-DEBREU securities.

- ▶ Notice that one can solve the equation for the ratio of marginal utilities for $c_t^i(s^t)$ as a function of:
 1. consumption of individual 1;
 2. the ratio of Lagrange multipliers of individuals i and 1

$$c_t^i(s^t) = u'^{-1} \left\{ u'[c_t^1(s^t)] \frac{\mu_i}{\mu_1} \right\}$$

- ▶ Substituting in the aggregate feasibility constraints gets :

$$\sum_i u'^{-1} \left\{ u'[c_t^1(s^t)] \frac{\mu_i}{\mu_1} \right\} = \sum_i y_t^i(s^t).$$

- ▶ This equation can be solved for $c_t^1(s^t)$.
- ▶ Notice that the right-hand-side is the aggregate endowment.

5. Time-0 trading: ARROW-DEBREU securities.

Proposition 2

The competitive equilibrium allocation is a function of the realized aggregate endowment and depends neither on the specific history leading up to that outcome nor on the realizations of individual endowments.

5. Time-0 trading: ARROW-DEBREU securities.

Pricing functions.

- ▶ Having found the equilibrium, we can use expressions such as:

$$\beta^t u'[c_t^i(s^t)] \pi_t(s^t) = \mu_i q_t^0(s^t)$$

to determine the equilibrium price $q_t^0(s^t)$.

- ▶ Notice that prices are stochastic processes.
- ▶ Price units are arbitrary so that we can normalize one price to unity.
- ▶ We set $q_0^0(s_0) = 1$.
- ▶ This implies $\mu_i = u'[c_0^i(s_0)]$.

5. Time-0 trading: ARROW-DEBREU securities. Equilibrium optimality.

- ▶ An important property of the competitive equilibrium is that it is PARETO optimal.
- ▶ This can be seen if set the PARETO weights in the Social Planner problem so that:

$$\lambda_i = 1/\mu_i, \quad \forall i$$

- ▶ $\theta_t(s^t) = q_t^0(s^t)$.
- ▶ The coincidence of competitive equilibria and PARETO optimality is behind the First and Second fundamental theorems of welfare economics.

5. Time-0 trading: ARROW-DEBREU securities. Equilibrium optimality.

Theorem 1

First Welfare Theorem: *Any Competitive Equilibrium is PARETO optimal.*

Theorem 2

Second Welfare Theorem: *Under some regularity conditions, any PARETO optimal allocation can be sustained as a competitive equilibrium.*

5. Time-0 trading: ARROW-DEBREU securities.

Equilibrium computation.

To compute the equilibrium one can use the NEGISHI *algorithm* to determine μ_i/μ_1 .

1. Fix an arbitrary (positive) value for μ_1 . Guess some initial positive value for the other μ_i . Compute:

$$c_t^i(s^t) = u'^{-1} \left\{ u' [c_t^1(s^t)] \frac{\mu_i}{\mu_1} \right\}.$$

$$\sum_i u'^{-1} \left\{ u' [c_t^1(s^t)] \frac{\mu_i}{\mu_1} \right\} = \sum_i y_t^i(s^t).$$

2. Solve for equilibrium prices $q_t^0(s^t)$:

$$q_t^0(s^t) = \beta^t u' [c_t^i(s^t)] \pi_t(s^t) \mu_i^{-1}$$

3. Check the budget constraint for every $i = 1, \dots, I$.
 - × For those i where c_i is too high, raise μ_i .
 - × For those i where c_i is too low, lower μ_i .
4. Iterate previous steps until convergence.

5. Time-0 trading: ARROW-DEBREU securities. Equilibrium computation.

- ▶ In general, the computation of equilibrium is a difficult problem.
- ▶ Typically, the equilibrium prices depend on the wealth distribution (*i.e.* all the individual endowment streams)
- ▶ There are some cases (particular preferences or endowment processes) where the computation is simpler, as we do not need to iterate on PARETO weights.

6. Simpler Computational Algorithm

6.1. Risk sharing

- ▶ The model we have described has important implications for risk sharing.
- ▶ Individuals with concave utility will want to smooth consumption over time.
- ▶ In the model we have studied, even without storage possibilities, smoothing is possible if individual shocks are not perfectly correlated.
- ▶ Indeed we will consider example where individuals can smooth quite a bit.

6. Simpler Computational Algorithm

6.1. Risk sharing

- ▶ Suppose utility is CRRA:

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0$$

- ▶ Then in the complete markets equilibrium:

$$\frac{[c_t^i(s^t)]^{-\gamma}}{\mu_i} = \frac{[c_t^j(s^t)]^{-\gamma}}{\mu_j}, \quad \forall i, j$$

$$c_t^i(s^t) = c_t^j(s^t) \left(\frac{\mu_i}{\mu_j} \right)^{-\frac{1}{\gamma}}, \quad \forall i, j$$

- ▶ Individual consumptions are perfectly correlated.

6. Simpler Computational Algorithm

6.1. Risk sharing

- ▶ The model implies that the consumption of different agents varies in the same proportion so that the ratio stays constant over time.
- ▶ There is extensive cross-period and cross-states risk sharing.
- ▶ The factor of proportionality is given by the ratio of multipliers μ_i/μ_j , or, alternatively, by the ratio of PARETO weights in the social planner program.
- ▶ From the FOC of the social planner program, we can notice that the only thing that determines individual consumption is aggregate shocks:

$$u'[c_t^i(s^t)]\lambda_i\beta^t\pi_t(s^t) = [c_t^i(s^t)]^{-\gamma}\lambda_i\beta^t\pi_t(s^t) = \theta_t(s^t).$$

6. Simpler Computational Algorithm

6.1. Risk sharing

- ▶ In this example, we can easily compute equilibrium prices. Using

$$c_t^i(s^t) = c_t^j(s^t) \left(\frac{\mu_i}{\mu_j} \right)^{-\frac{1}{\gamma}}, \quad \forall i, j$$

- ▶ and the price formula derived earlier

$$\beta^t u'[c_t^i(s^t)] \pi_t(s^t) = \mu_i q_t^0(s^t)$$

- ▶ we obtain

$$q_t^0(s^t) = \mu_i^{-1} \alpha_i^{-\gamma} \beta^t (\bar{y}_t(s^t))^{-\gamma} \pi_t(s^t)$$

where $c_t^i(s^t) = \alpha_i \bar{y}_t(s^t)$, $\bar{y}_t(s^t) = \sum_i y_t^i(s^t)$, $\alpha_i = \left(\sum_i \left(\frac{\mu_i}{\mu_j} \right)^{\frac{1}{\gamma}} \right)^{-1}$

- ▶ We have one normalisation choice, which amount to setting $\mu_i^{-1} \alpha_i^{-\gamma}$ for one i to an arbitrary positive number.
- ▶ For example $\mu_1^{-1} \alpha_1^{-\gamma} = 1$

6. Simpler Computational Algorithm

6.1. Risk sharing

- ▶ We then compute equilibrium as follows:
- ▶ Prices are obtained from

$$q_t^0(s^t) = \beta^t (\bar{y}_t(s^t))^{-\gamma} \pi_t(s^t)$$

- ▶ Using those prices and consumer's i budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

- ▶ We obtain the consumption share α^i as its share of total wealth evaluated at equilibrium prices:

$$\alpha^i = \frac{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)}{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \bar{y}_t(s^t)}$$

6. Simpler Computational Algorithm

6.1. Risk sharing

Empirical implications

- ▶ Using the Social planner FOC

- ▶ We have

$$u'[c_t^i(s^t)]\lambda_i\beta^t\pi_t(s^t) = [c_t^i(s^t)]^{-\gamma}\lambda_i\beta^t\pi_t(s^t) = \theta_t(s^t).$$

- ▶ Taking logs:

$$-\gamma\log(c_t^i(s^t)) + \log(\lambda_i) + \log(\beta^t\pi_t(s^t)) = \log(\theta_t(s^t)).$$

$$-\gamma\log(c_t^i(s^t)) + \log(\lambda_i) = \log(\theta_t(s^t)) - t\log(\beta) - \log(\pi_t(s^t)).$$

- ▶ Taking this equation at two different points in time and subtracting one from the other:

$$\begin{aligned} -\gamma[\log(c_t^i(s^t)) - \log(c_\tau^i(s^\tau))] &= \log(\theta_t(s^t)) + \log(\pi_t(s^t)) - \log(\theta_\tau(s^\tau)) \\ &\quad - \log(\pi_\tau(s^\tau)) - (t - \tau)\log(\beta) = \nu_{t,\tau}. \end{aligned}$$

- ▶ No index i in $\nu_{t,\tau}$.

6. Simpler Computational Algorithm

6.1. Risk sharing

Empirical implications

- ▶ This equation gives an empirical prediction:

$$[\log(c_t^i(s^t)) - \log(c_\tau^i(s^\tau))] = -\frac{1}{\gamma} \nu_{t,\tau}$$

- ▶ Note:
 - × Strong empirical test: individual consumption growth only depends on aggregate growth.
 - × Townsend (1994) and others used this strategy explicitly.
 - × There are no residuals in this equation
 - ▶ Measurement error can be added.
 - ▶ Taste shocks.

6. Simpler Computational Algorithm

6.2. Other examples

- ▶ There are other examples in Chapter 8 of LJUNQVIST & SARGENT's book that you have to work by yourself.

7. Primer on Asset pricing

- ▶ Many asset-pricing models assume complete markets and price an asset by breaking it into a sequence of history-contingent claims, evaluating each component of that sequence with the relevant “state price deflator” $q_t^0(s^t)$ then adding up those values.
- ▶ The asset is viewed as redundant, in the sense that it offers a bundle of history-contingent dated claims, each component of which has already been priced by the market.

7. Primer on Asset pricing

7.1. Pricing Redundant Assets

- ▶ Let $\{d(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t , state s^t consumption, where $d(s_t)$ is a measurable function of s_t .
- ▶ The price of an asset paying the owner this stream must be

$$p_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d(s^t)$$

- ▶ This can be understood as an arbitrage equation.

7. Primer on Asset pricing

7.2. Riskless Consol

- ▶ A riskless consol offers for sure one unit of consumption at each period, i.e. $d_t(s^t) = 1$ for all t and s^t .
- ▶ The price is

$$p_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)$$

7. Primer on Asset pricing

7.3. Riskless strips

- ▶ Consider a sequence of strips of returns on the riskless consol.
- ▶ The time- t strip is the return process $d_\tau = 1$ if $\tau = t \geq 0$, and 0 otherwise.
- ▶ The price of time- t strip at 0 is

$$p_t^0 = \sum_{s^t} q_t^0(s^t)$$

7. Primer on Asset pricing

7.4. Tail assets

- ▶ Consider the stream of dividends $\{d(s_t)\}_{t \geq 0}$
- ▶ For $\tau \geq 1$, suppose that we strip off the first $\tau - 1$ periods of the dividend and want to get the time-0 value of the dividend stream $\{d(s_t)\}_{t \geq \tau}$.
- ▶ Let $p_\tau^0(s^\tau)$ be the time-0 price of an asset that entitles the dividend stream $\{d(s_t)\}_{t \geq \tau}$ if history s^τ is realized:

$$p_\tau^0(s^\tau) = \sum_{t \geq \tau} \sum_{\{\tilde{s}^t: \tilde{s}^\tau = s^\tau\}} q_t^0(\tilde{s}^t) d(\tilde{s}^t)$$

- ▶ Let us convert this price into units of time τ , state s^τ by dividing by $q_\tau^0(s^\tau)$:

$$p_\tau^\tau(s^\tau) = \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t \geq \tau} \sum_{\{\tilde{s}^t: \tilde{s}^\tau = s^\tau\}} \frac{q_t^0(\tilde{s}^t)}{q_\tau^0(s^\tau)} d(\tilde{s}^t)$$

- ▶ Notice that for all consumers i

$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u'[c_t^i(s^t)] \pi(s^t)}{\beta^\tau u'[c_\tau^i(s^\tau)] \pi(s^\tau)} = \beta^{t-\tau} \frac{u'[c_t^i(s^t)]}{u'[c_\tau^i(s^\tau)]} \pi(s^t | s^\tau) \quad (\star)$$

7. Primer on Asset pricing

7.4. Tail assets

- ▶ $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t , state s^t in terms of the date- τ , state- s^τ consumption good.
- ▶ The price at τ in history s^τ for the tail asset is

$$p_\tau^\tau(s^\tau) = \sum_{t \geq \tau} \sum_{\{\tilde{s}^t: \tilde{s}^\tau = s^\tau\}} q_t^\tau(\tilde{s}^t) d(\tilde{s}^t)$$

- ▶ This tail asset formula is useful if one wants to create in a model a time series of equity prices: an equity purchased at time τ entitles the owner to the dividends from time τ forward, and the price is given by the above formula.

7. Primer on Asset pricing

7.4. Tail assets

$$q_t^\tau(s^t) = \beta^{t-\tau} \frac{u'[c_t^i(s^t)]}{u'[c_\tau^i(s^\tau)]} \pi(s^t | s^\tau) \quad (\star)$$

- ▶ We have shown that $c_t^i(s^t)$ are not history dependant.
- ▶ Then the relative price in (\star) is not history. dependent.

Proposition 3

The equilibrium price of date- $t \geq 0$, state- s^t consumption good expressed in terms of date τ ($0 \leq \tau \leq t$), state s^τ consumption good is not history dependent:

$$q_t^\tau(s^t) = q_t^j(\tilde{s}^k)$$

for $j, k \geq 0$ such that $t - \tau = k - j$ and $[s_t, s_{t-1}, \dots, s_\tau] = [\tilde{s}_k, \tilde{s}_{k-1}, \dots, \tilde{s}_j]$.

7. Primer on Asset pricing

7.5. Pricing One Period Returns

- ▶ The one-period version of equation (★) is

$$q_{\tau+1}^{\tau}(s^{\tau+1}) = \beta \frac{u'(c_{\tau+1}^i)}{u'(c_{\tau}^i)} \pi(s_{\tau+1}|s_{\tau})$$

- ▶ The RHS is the one-period *pricing kernel* at time τ .
- ▶ The price at time τ in state s^{τ} of a claim to a random payoff $\omega(s_{\tau+1})$ is given, using the pricing kernel, by

$$\begin{aligned} p_{\tau}^{\tau}(s^{\tau}) &= \sum_{s_{\tau+1}} q_{\tau+1}^{\tau}(s^{\tau+1}) \omega(s_{\tau+1}) \\ &= E_{\tau} \left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} \omega(s_{\tau+1}) \right] \end{aligned}$$

where superscripts i and dependence to s_{τ} have been deleted.

- ▶ Let denote the one-period gross return on the asset by $R_{\tau+1} = \omega(s_{\tau+1})/p_{\tau}^{\tau}(s^{\tau})$. Then, for any asset, the above equation implies

$$1 = E_{\tau} \left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} R_{\tau+1} \right]$$

7. Primer on Asset pricing

7.5. Pricing One Period Returns

$$1 = E_{\tau} \left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} R_{\tau+1} \right]$$

- ▶ The term $m_{\tau+1} = \beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})}$ is a *stochastic discount factor*.
- ▶ **That equation** can be understood as a restriction on the conditional moments of returns and $m_{\tau+1}$.
- ▶ Applying the law of iterated expectations to the above equation, one gets the unconditional moments restriction:

$$1 = E \left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} R_{\tau+1} \right]$$

8. Sequential trading.

- ▶ After considering the time-0 trading we consider sequential trading.
- ▶ For this, the one-period formula we derived in the previous lecture will be crucial.
- ▶ We will show that the allocations that prevail under complete markets, as described by the zero-period trade in ARROW-DEBREU state contingent assets, can be replicated by one-period securities.
- ▶ We will consider the household value function and write it as a function of a crucial state variable.

8. Sequential trading.

- ▶ We define as 'household wealth' the value (at a point in time) of all the claims 'owned' by a household net of its liabilities.

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) [c_\tau^i(s^\tau) - y_\tau^i(s^\tau)].$$

- ▶ Notice that at time t , given history s^t , we can ignore all the claims and liabilities that do not correspond to that particular history.
- ▶ feasibility implies

$$\sum_{i=1}^I \Upsilon_t^i(s^t) = 0, \quad \forall t, s^t.$$

8. Sequential trading.

Debt limits.

- ▶ When considering the time-0 equilibrium we impose the intertemporal budget constraint.
- ▶ In the case of sequential equilibria with infinitely lived consumers we need to avoid 'Ponzi schemes'.
- ▶ This is equivalent to assuming the consumer will have to repay her debts in any state of the world.
- ▶ We will impose a *natural debt limit*, given, for every history, by the sum of future endowments.

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau).$$

- ▶ Each consumer will not be able to promise to pay, for each history s^t more than $A_t^i(s^t)$, which corresponds to zero consumption.

8. Sequential trading.

Debt limits.

- ▶ Suppose our consumers are operating in a sequence of markets in one-period-ahead state-contingent claims.
 - × At time t consumers trade on claims contingent on s_{t+1} .
- ▶ Let $\tilde{a}_t^i(s^t)$ be claims brought into time t .
- ▶ $\tilde{Q}_t(s_{t+1}|s^t)$ is the price (at time t) of one unit of time $t + 1$ - consumption, contingent on s_{t+1} , when history has been s^t .
- ▶ Intertemporal budget constraint:

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t).$$

- ▶ Debt limits:

$$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}).$$

8. Sequential trading.

Consumer Problem

- The Lagrangian for the consumer problem will be given by:

$$L^i = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \beta^t u(\tilde{c}_t^i(s^t)) \pi_t(s^t) \right. \\ \left. + \eta_t^i(s^t) \left[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1} | s^t) \right] \right. \\ \left. + \sum_{s_{t+1}} \tilde{\nu}_t^i(s_{t+1}, s^t) \left[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1}) \right] \right\}$$

for a given initial wealth $\tilde{a}_0^i(s_0)$.

8. Sequential trading.

First Order Conditions.

- ▶ First Order Conditions w.r.t. $\tilde{c}_t^i(s^t)$ and $\tilde{a}_{t+1}^i(s_{t+1}, s^t)$:

$$\beta^t u'(\tilde{c}_t^i(s^t)) \pi_t(s^t) - \eta_t^i(s^t) = 0.$$

$$-\eta_t^i(s^t) \tilde{Q}_t(s_{t+1}|s^t) + \tilde{v}_t^i(s_{t+1}, s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

- ▶ Because of the Inada conditions, the debt limit constraint will not be binding at the optimum.

$$\implies \tilde{v}_t^i(s_{t+1}, s^t) = 0, \quad \forall t, s^t, s_{t+1}.$$

$$-\eta_t^i(s^t) \tilde{Q}_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

8. Sequential trading.

First Order Conditions.



$$\beta^t u'(\tilde{c}_t^i(s^t)) \pi_t(s^t) - \eta_t^i(s^t) = 0.$$

$$-\eta_t^i(s^t) \tilde{Q}_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

▶ Substituting the first equation in the second yields:

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\tilde{c}_{t+1}^i(s^{t+1}))}{u'(\tilde{c}_t^i(s^t))} \pi_t(s_{t+1}|s^t), \quad \forall t, s^t, s_{t+1}.$$

8. Sequential trading.

Definition 8

A **distribution of wealth** is a vector $\vec{a}_t(s^t) = \{\tilde{a}_t^i(s^t)\}_{i=1}^I$ such that $\sum_i \tilde{a}_t^i(s^t) = 0$.

Definition 9

A **sequential-trading competitive equilibrium** is an initial distribution of wealth $\vec{a}_0(s_0)$, an allocation $\{\tilde{c}^i\}_{i=1}^I$, and pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ such that:

1. for all i , given $\vec{a}_0(s_0)$ and the pricing kernels, the consumption allocation \tilde{c}^i solves household i consumption problem.
2. for all realizations of $\{s^t\}_{t=0}^\infty$ the households' consumption allocation and implied asset portfolios satisfy:

$$\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t),$$

$$\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0.$$

9. Equivalence of allocations under time-zero trading and sequential trading.

- ▶ With an appropriate choice of pricing kernel, one can show that a competitive equilibrium allocation of the complete markets model with time 0 trading is also a sequential-trading competitive equilibrium allocation ...
- ▶ ..., an equilibrium for a particular initial distribution of wealth

9. Equivalence of allocations under time-zero trading and sequential trading.

Pricing kernel choice

- ▶ Consider the price $q_t^0(s^t)$ in the ARROW-DEBREU equilibrium.
- ▶ Consider the sequence of pricing kernels given by:

$$\tilde{Q}_t(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_{t+1}^t(s^{t+1}).$$

- ▶ Consider the ARROW-DEBREU equilibrium consumption allocation $\{c_t^i(s^t)\}$.
- ▶ Given the pricing kernels just defined, it follows that:

$$\frac{\beta u'[c_{t+1}^i(s^{t+1})] \pi(s^{t+1}|s^t)}{u'[c_t^i(s^t)]} = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \tilde{Q}_t(s_{t+1}|s^t)$$

- ▶ It follows that the ARROW-DEBREU time-0 trading allocation satisfies the Euler equation (first order condition) for the sequential trading problem when the pricing kernel are those defined.

9. Equivalence of allocations under time-zero trading and sequential trading.

Initial wealth distribution

- ▶ Sequential trading allocations are indexed by the initial wealth distribution.
- ▶ We therefore need to choose a wealth distribution that generates the *ARROW-DEBREU* allocation.
- ▶ We conjecture that the initial wealth distribution is the null vector.
- ▶ Using the intertemporal budget constraints we prove that the portfolio choices induced imply the same sequence of consumption and that this is optimal.

9. Equivalence of allocations under time-zero trading and sequential trading.

Initial wealth distribution

- Suppose that at time t and history s^t , household i chooses the following asset portfolio :

$$\tilde{a}_{t+1}^i(s_{t+1}, s^t) = \Upsilon_{t+1}^i(s^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q_\tau^{t+1}(s^\tau) [c_\tau^i(s^\tau) - y_\tau^i(s^\tau)].$$

where the consumption sequence is the ARROW-DEBREU equilibrium one.

- The value of that portfolio is

$$\begin{aligned} \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1} | s^t) &= \sum_{s^{t+1} | s^t} \Upsilon_{t+1}^i(s^{t+1}) q_{t+1}^t(s^{t+1}) \\ &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} [c_\tau^i(s^\tau) - y_\tau^i(s^\tau)] q_\tau^t(s^\tau). \end{aligned}$$

- Notice $q_\tau^{t+1}(s^\tau) q_{t+1}^t(s^{t+1}) = \frac{q_\tau^0(s^\tau)}{q_{t+1}^0(s^{t+1})} \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_\tau^t(s^\tau)$, ($\tau > t$).

9. Equivalence of allocations under time-zero trading and sequential trading.

- ▶ Given an initial wealth (0), a realization for the endowments and the choice of assets, we have consumption $\tilde{c}_\tau^i(s^\tau)$.
- ▶ Starting with time 0:

$$\tilde{c}_0^i(s^0) + \sum_{\tau=1}^{\infty} \sum_{s^\tau} [c_t^i(s^t) - y_t^i(s^t)] q_t^0(s^t) = y_0^i(s^0) + 0$$
$$\implies \tilde{c}_0^i(s^0) = c_0^i(s^0).$$

- ▶ Therefore, the proposed portfolio strategy attains the same consumption plan as in the competitive equilibrium of the **ARROW-DEBREU** economy
- ▶ But is that the best choice for agents?
- ▶ Yes, as the natural debt limit precludes to choose a consumption plan with higher utility.