University College London, 2019-2020
Econ 0039 – Advanced Macro – Franck Portier

Problem set 1: (very) Long Run growth
Version 1.0 - 06/01/2020
Partial Solution

Problem III – The Malthusian Regime

The model is in discrete time. Consider the following model of joint determination of population and income per capita. The birth rate is given by:

$$b_t = \alpha_b + \beta_b y_t, \tag{1}$$

the mortality rate by:

$$m_t = \alpha_m - \beta_m y_t, \tag{2}$$

and the production function is:

$$Y_t = \alpha_y + \beta_y P_t. \tag{3}$$

Y stands for total production (or income), P for population and y for income per capita. It is assumed that all the population works and that there is non immigration nor emigration. All coefficients of the model are positive. It is assumed that $\alpha_m > \alpha_b$ and $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$.

1 – Discuss equations (1) to (3). Compute marginal and average productivity of labor. Comment.

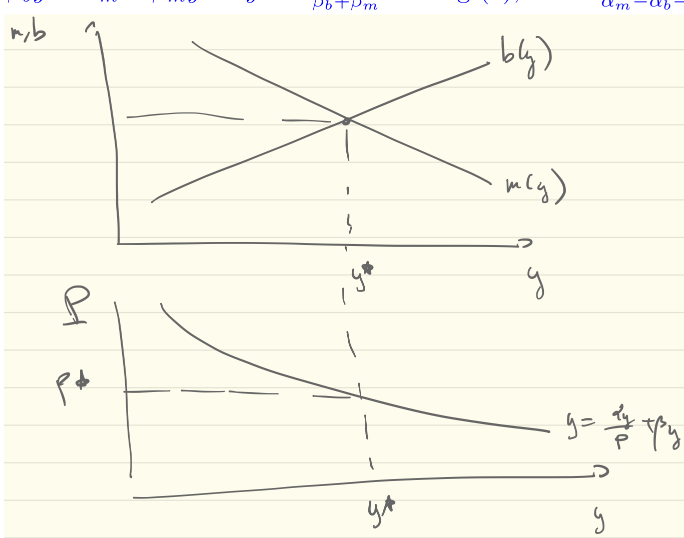
See slides for discussion. Average productivity is

$$y_t = \frac{Y_t}{P_t} = \frac{\alpha_y}{P_t} + \beta_y. \tag{4}$$

Marginal productivity is $\frac{dY_t}{dP_t} = \beta_y$. Average productivity is decreasing, marginal productivity is constant.

2 – Compute the steady state level of total income, per capita income and population. Show graphically how those steady state values are determined.

(1) and (2) imply $P_{t+1} = P_t + b_t P_t - m_t P_t = (1 + b_t - m_t$. The steady state (in which population is constant) is obtained for $b_t = m_t \Leftrightarrow \alpha_b + \beta_b y = \alpha_m - \beta_m y \Leftrightarrow y^\star = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m}$. Using (4), $P^\star = \frac{\alpha_y(\beta_y + \beta_m)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}$.



3 – For the rest of the exercice, we assume $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = \alpha$. What are the steady state levels for this configuration of parameters?

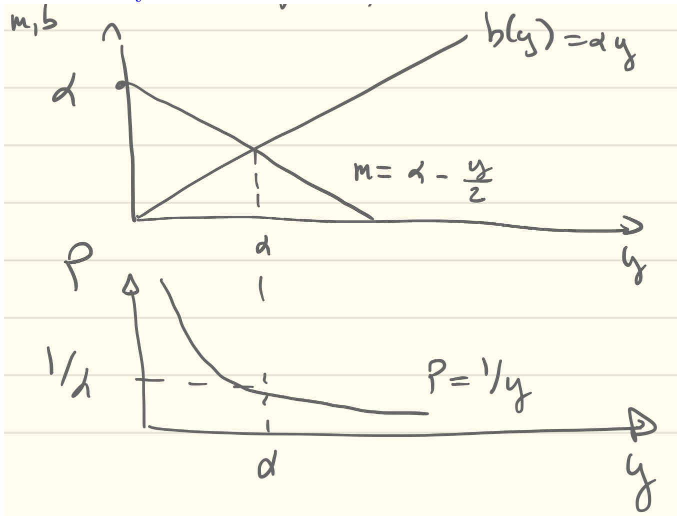
We have in that case $y^\star = \alpha$ and $P^\star = 1/\alpha$.

4 – Show that the model dynamics can be summarized by a first order difference equation in P_t (of the type $P_{t+1} = f(P_t)$).

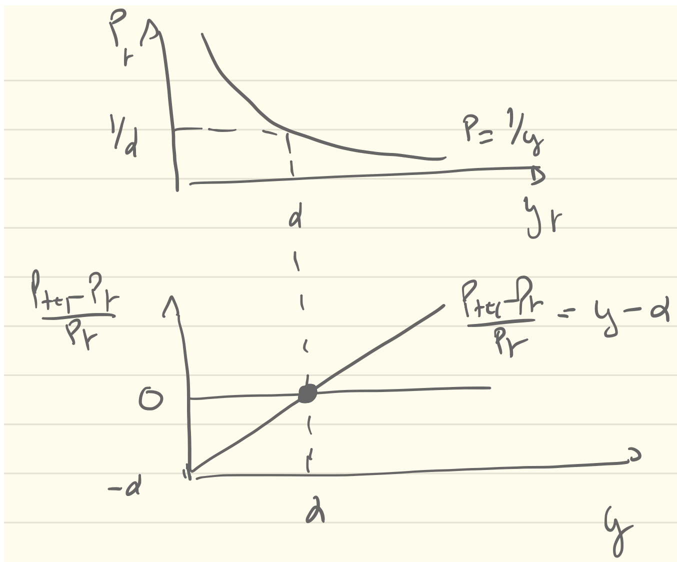
The dynamics of the model is given by $P_{t+1} = (1 + b_t - m_t)P_t$, which gives

$$\frac{P_{t+1} - P_t}{P_t} = y_t - \alpha. \tag{5}$$

The graphical determination if the steady state is obtained as follows:



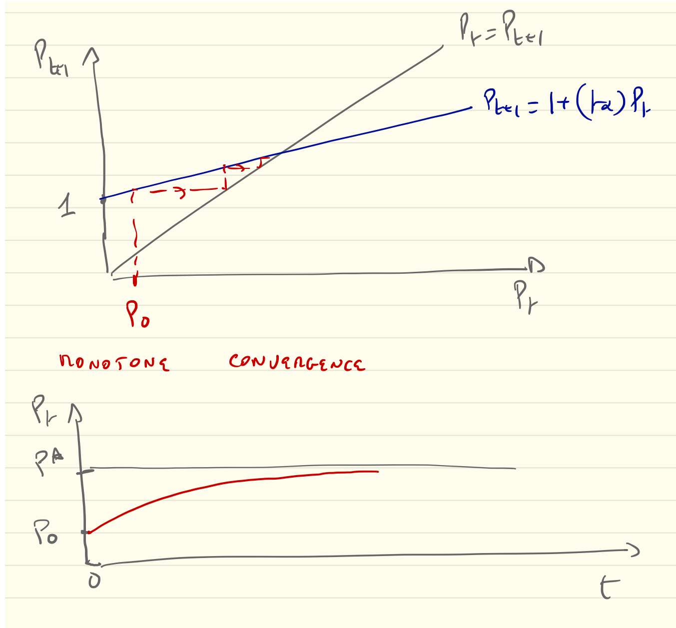
and the dynamics is obtained as follows:



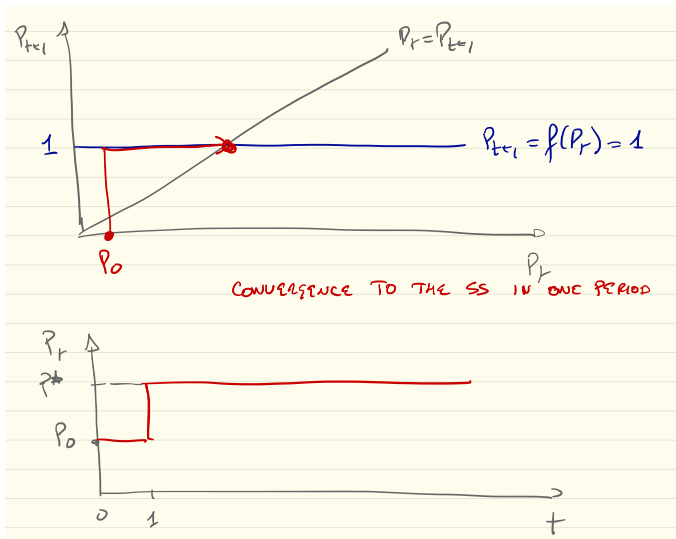
5 – Study the convergence of population to its the steady state starting from a P_0 close to 0 for the following values of α: *(i)* $0 < \alpha < 1$, *(ii)* $\alpha = 1$, *(iii)* $1 < \alpha < 2$, *(iv)* $\alpha = 2$, *(v)* $\alpha > 2$.

Let's start from a P_0 close to zero to fix ideas.

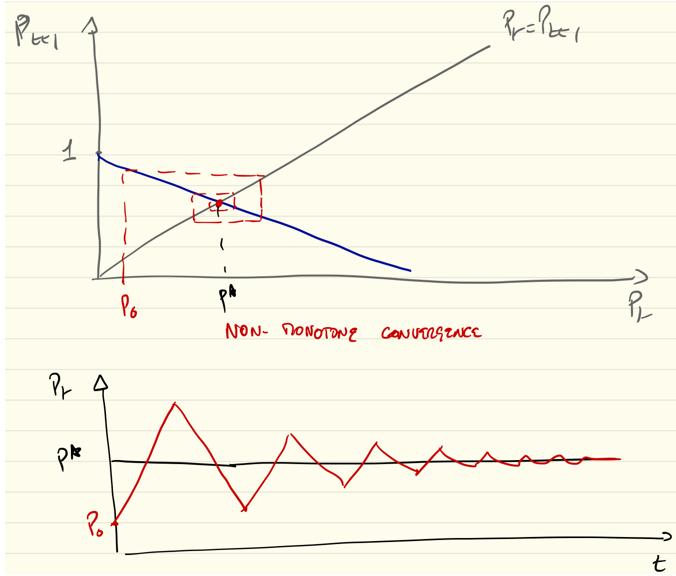
- for $0 < \alpha < 1$



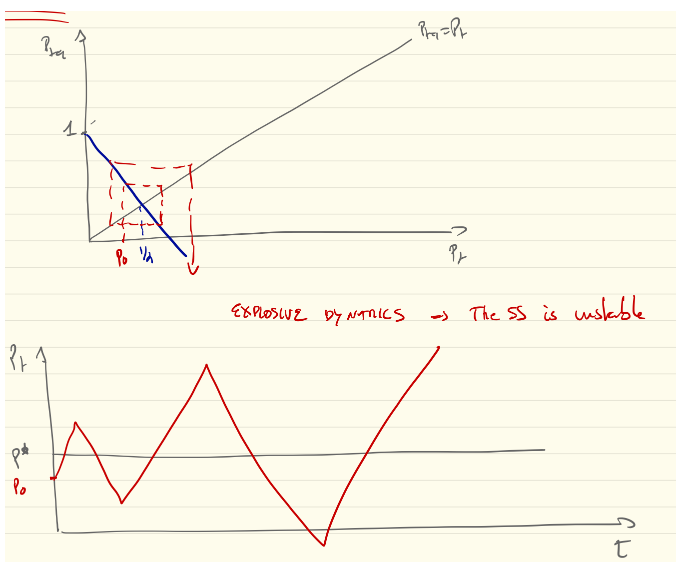
- for $\alpha = 1$



- for $1 < \alpha < 2$



- for $\alpha > 2$



Problem IV – Modeling mortality

We assume that mortality i.e. modeled as a rare event in continuous time. To do so, we make use of the exponential distribution. More precisely, we assume that life length is a continuous stochastic variable, whose probability density function is given by

$$f(L) = me^{-mL}$$

where $L \geq 0$ is life length and m is a positive parameter that we refers to as the mortality rate. The unit of time is the year and $m = 0.01$.

1 – Compute the cumulative distribution function of L, defined as $F(L) = P(L \leq \ell)$.

$$P(L > \ell) = \int_{\ell}^{\infty} me^{-mL} dL = m\left[-\frac{1}{m}e^{-mL}\right]_{\ell}^{\infty} = e^{-m\ell},$$

so that $F(L) = 1 - e^{-m\ell}$.

2 – Compute $P(L > \ell)$.
$P(L > \ell) = e^{-m\ell}$.

3 – Compute the probability that life lasts at least 10 years.
This is $P(L > 10) = e^{-.01\times 10} = .9048$

4 – Compute the probability that life lasts at least 80 years.
This is $P(L > 80) = e^{-.01\times 80} = .4493$

5 – Compute the probability for a person of age 10 that she lives at least until the age of 20. [Hint: use the Bayes formula $P(A|B) = P(A \cap B)/P(B)$]. Compare the result to the answer top question 3. Why is this model often called the "perpetual youth" model?

$$P(L > 20|L > 10) = \frac{P(L > 20 \text{ and } L > 10)}{P(L > 10))} = \frac{P(L > 20)}{P(L > 10)} = \frac{e^{-.01\times 20}}{e^{-.01\times 10}} = \underbrace{e^{-.01\times 20}}_{A} = .9048.$$

Note that A is the probability of living another 10 years. It is independent of age (compare to question 3), hence the name of "perpetual youth" model.

6 – Prove more generally the property of memorylessness of the distribution, *i.e.* $P(L > a + b|L > a) = P(L > b)$.
We have $P(L > b) = e^{-mb}$. Then,

$$P(L > a + b|L > a) = \frac{P(L > a + b \text{ and } L > a)}{P(L > a))} = \frac{P(L > a + b)}{P(L > a)} = \frac{e^{-m(a+b)}}{e^{-ma}} = e^{-mb} = P(L > b).$$

7 – Compute expected life length at birth and at age 50. Discuss.

Life expectancy at birth $\mathcal{E}_0$ is (see slides)

$$\mathcal{E}_0 = \int_0^\infty Lf(L)dL = \int_0^\infty LmLe^{-mL}dL = \frac{1}{m} = 100 \text{ years.}$$

Because the process is memoryless, expected life left at age 50 $\mathcal{E}_{50}$ is equal to $\mathcal{E}_0$. Therefore, expected age at death when aged 50 is 150 years.

Problem V – A model of unified growth in which adults consume food

We have studied in section 6 of Lecture 1 a model of unified growth. One unrealistic model assumption was that adults do not consume food, but only buy food for there children. In this problem, we remove this assumption and compute again the model equilibrium.

Adults like manufacturing goods m_t, adult food f_t and children n_t. Their utility is given by

$$u_t = m_t + \gamma \log n_t + \delta \log f_t.$$

As in the lecture, the manufacturing good is the numéra ire and one children costs one unit of food, so that the budget constraint of an adult in t is :

$$p_t n_t + p_t f_t + m_t = w_t.$$

The solution to that model is very similar to the one we have studied in class.

1 – Solve the utility maximization problem of an adult and derive the demand for children and the demand for agricultural goods (food for chidden and for adults).
We need to solve

$$\begin{aligned} \max_{n_t, m_t, f_t} \quad & m_t + \gamma \log n_t + \delta \log f_t \\ s.t. \quad & p_t(n_t + f_t) + m_t = w_t \end{aligned}$$

One can check that first order conditions give $n_t = \frac{\gamma}{p_t}$ and $f_t = \frac{\delta}{p_t}$. Nite that we have $f_t = \frac{\delta}{\gamma} n_t$.

2 – Show that total demand for food is $\frac{(\gamma+\delta)}{p_t} L_t$.
Total demand for food is $(n_t + f_t)L_t = \frac{(\gamma+\delta)}{p_t} L_t$

As in the lecture, technology is for food $Y_t^A = A_t^\epsilon (L_t^A)^\alpha X^{1-\alpha}$, where X is land, assumed to be in fixed supply, and normalized to $X = 1$; L^A is agricultural labor and A is productivity on agriculture. ϵ and α are between 0 and 1. The technology for the manufacturing good is $Y_t^M = M_t^\epsilon L_t^M$. Note that we assume for simplicity that land is free disposal (land rent is zero). We finally assume Learning-By-Doing, so that $A_{t+1} - A_t = Y_t^A$ and $M_{t+1} - M_t = Y_t^M$.

3 – Write the equilibrium condition on the food market and derive from that condition an expression for the share of agricultural labor in total labor $\theta_t = \frac{L_t^A}{L_t}$. How is this expression different from the one in the lecture? Discuss.

It is straightforward to obtain

$$\theta_t = \left(\frac{\left(\frac{\gamma+\delta}{\gamma}\right) n_t L_t^{1-\alpha}}{A_t^{\epsilon}} \right)^{\frac{1}{\alpha}}$$

The only difference with the slides is the term $\left(\frac{\gamma+\delta}{\gamma}\right)$, which is 1 if $\delta = 0$, which is indeed the model of the slides)

4 – Give an expression for the wages in the two sectors.

$w_t^A = \frac{p_t Y_t^A}{L_t^A}$ and $w_t^M = \frac{Y_t^M}{L_t^M}$

5 – Why free movements of workers between sectors will equalize the wages? From the wage equality condition, derive an expression for p_t as a function of M_t, A_t, θ_t and L_t.

$$p_t = \left(\frac{\gamma+\delta}{\gamma}\right)^{\frac{1-\alpha}{xa}} \frac{M_t^{\alpha\epsilon}(\gamma L_t)^{1-\alpha}}{A_t^{\epsilon}}$$

6 – Using the demand for children, derive n_t as a function of M_t, A_t, and L_t. How is this expression different from the one in the lecture? Discuss.

$$n_t = \gamma \frac{A_t^{\epsilon}}{M_t^{\alpha\epsilon}(\gamma+\delta)L_t^{1-\alpha}}$$

As adults also demand food for themselves when $\delta > 0$, there is less food left for children, and hence less children per adult. Fertility is lower than in the model we studies in class (if parameters are not changed).

7 – Show that the long run level of population growth is the same than in the model of Lecture 5.

When computing $\frac{n_{t+1}}{n_t}$, the term $(\gamma + \delta)$ cancels out, so that nothing is changed as compared to the slides.

Problem VII – Dynamics and Stagnation in the Malthusian Epoch

This problem follows Ashraf & Galor, The American Economic Review, Vol. 101, No. 5, 2011) and proposes a model of the Malthusian regime. I suggest that you read that paper, which is in the appendix to this problem set.

Consider an overlapping-generations economy in which activity extends over infinite discrete time. In every period, the economy produces a single homogeneous good using land and labor as inputs. The supply of land is exogenous and fixed over time, whereas the evolution of labor supply is governed by households' decisions in the preceding period regarding the number of their children.

Production occurs according to a constant-returns-to-scale technology. The output produced at time t, Y_t is

$$Y_t = (AX)^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where L, and X are, respectively, labor and land employed in production in period t, and A measures the technological level. The technological level may capture the percentage of arable land, soil quality, climate, cultivation and irrigation methods, as well as the knowledge required for engagement in agriculture (i.e., domestication of plants and animals). Thus, AX captures the effective resources used. Output per worker produced at time t is denoted y_t.

In each period t, a generation consisting of L_t identical individuals joins the workforce. Each individual has a single parent. Members of generation t live for two periods. In the first period of life (childhood), $t-1$, they are supported by their parents. In the second period of life (parenthood), t, they inelastically supply their labor (one unit), generating an income that is equal to the output per worker, y_t, which they allocate between their own consumption and that of their children. Individuals generate utility from consumption and the number of their children:

$$u^t = c_t^{1-\gamma} n_t^{\gamma} \qquad 0 < \gamma < 1,$$

where c_t is consumption and n_t, is the number of children of an individual of generation t.

Members of generation t allocate their income between their consumption, c_t and expenditure on children, ρn_t, where ρ is the cost of raising a child. Hence, the budget constraint for a member of generation t (in the second period of life) is

$$\rho n_t + c_t \leq y_t.$$

1 – Write the utility maximisation problem of an adult and derive optimal consumption c_t and fertility n_t as a function of income per capita. What is the impact of income on fertility? Comment.

An adult solves the following problem:

$$\max_{c_t, n_t} c_t^{1-\gamma} n_t^{\gamma}$$

subject to

$$\rho n_t + c_t \leq y_t.$$

Solving FOC gives

$$\begin{aligned} c_t &= (1-\gamma) y_t, \\ n_t &= \frac{\gamma}{\rho} y_t. \end{aligned}$$

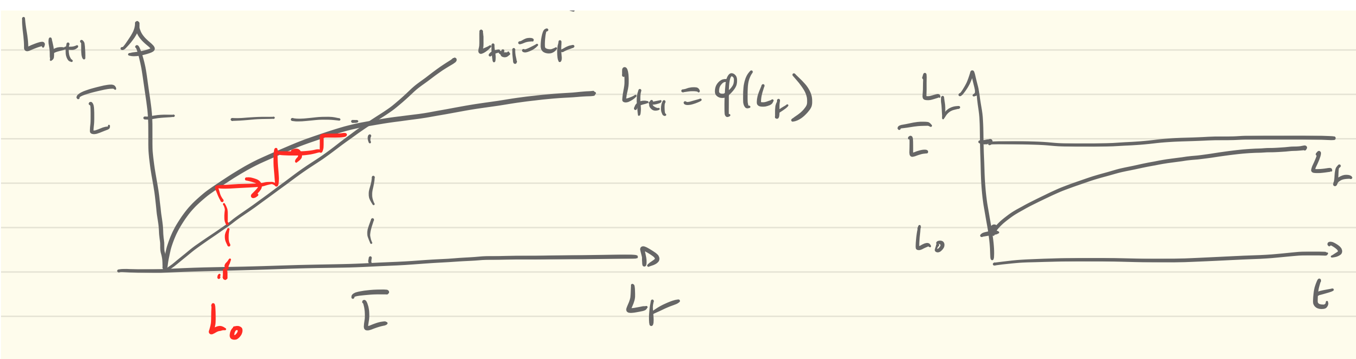
We find that fertility is an increasing function of income, which is well in line with the mechanics of the Malthusian regime.

2 – Consider that the initial size of the working population is $L_0 > 0$. Explain why the law of motion of adult population is $L_{t+1} = n_t L_t$. Use the result of the previous question to derive the equilibrium law of motion of adult population $L_{t+1} = \phi(L_t)$.

In periods t, each adult has n_t children (note that the (unrealistic) assumption in that model is that there is asexual reproduction, as one adult alone can have children). As there are L_t adults, there are $n_t L_t$ children today, and therefore $L_{t+1} = n_t L_t$ adults tomorrow.
Using $n_t = \frac{\gamma}{\rho} y_t$ and $y_t L_t = Y_t$, we obtain $L_{t+1} = \frac{\gamma}{\rho} Y_t$. Using the production function, we then have

$$L_{t+1} = \frac{\gamma}{\rho} (AX)^{\alpha} L_t^{1-\alpha} = \phi(L_t).$$

3 – Show graphically in the plane (L_t, L_{t+1}) that L will converge to a steady state. Compute that steady state $\overline{L}$. Draw the time path of L starting from $L_0 < \overline{L}$.



The steady state is defines by $\overline{L} = \phi(\overline{L})$, that gives

$$\overline{L} = \left(\frac{\gamma}{\rho}\right)^{\frac{1}{\alpha}} AX.$$

4 – Compute steady state population density $\overline{P}_d$.

Steady state population density is

$$\overline{P}_d = \frac{\overline{L}}{X} = \left(\frac{\gamma}{\rho}\right)^{\frac{1}{\alpha}} A$$

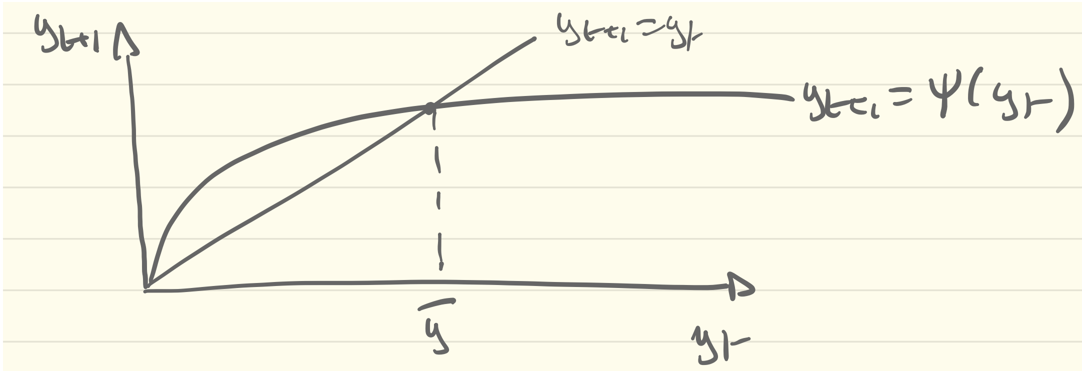
5 – Compute the law of motion of output per worker $y_{t+1} = \psi(y_t)$. Compute the steady state level of y, denoted $\overline{y}$. Interpret the result.

Output per worker dynamics is given by

$$y_{t+1} = \frac{Y_{t+1}}{L_{t+1}} = (AX)^{\alpha} (n_t L_t)^{-\alpha} = \left(\frac{AX}{L_t}\right)^{\alpha} n_t^{-\alpha} = \frac{y_t}{n_t^{\alpha}}.$$

Using the expression for n_t, we obtain

$$y_t = \left(\frac{\rho}{\gamma}\right)^{\alpha} y_t^{1-\alpha} = \psi(y_t).$$



The steady state level of y is therefore

$$\overline{y} = \frac{\rho}{\gamma}.$$

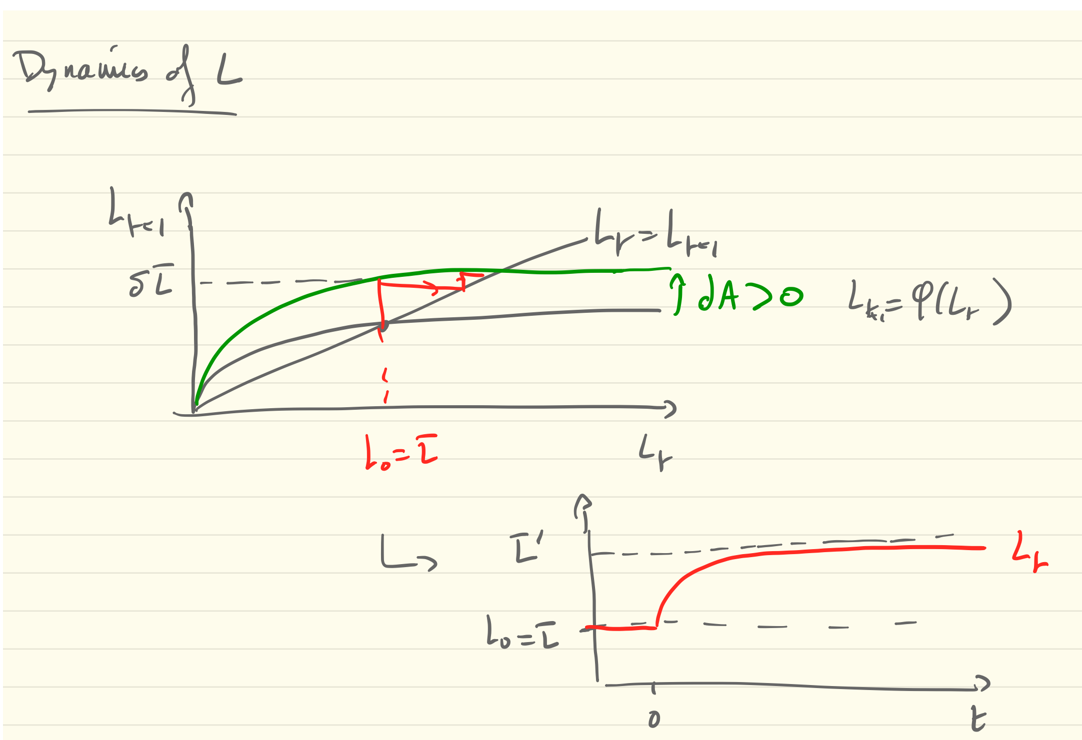
$\overline{y}$ depends positively on the cost of raising a child and negatively on the utility of children.

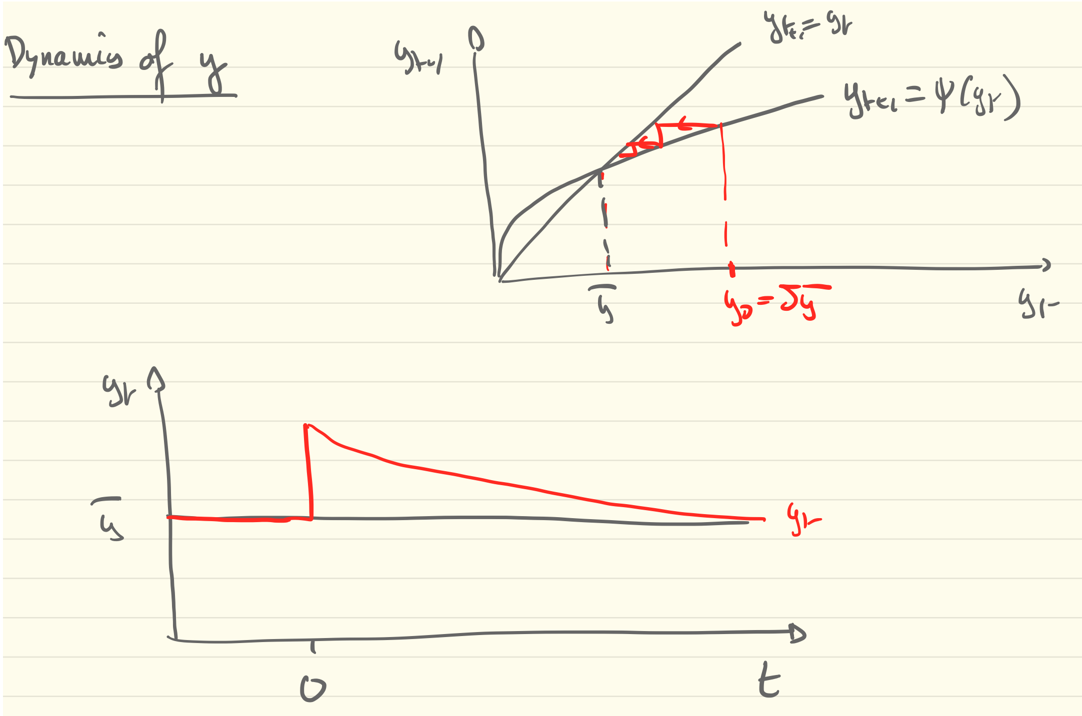
6 – Assume that the L_0 is at its steady state level $\overline{L}$ and that in period 0 technology A permanently increases to the level δA, $\delta > 1$. Compute y_0, n_0 and L_1. Draw the time path of adult population L_t and output per worker y_t following that increase in A.

The economy starts from δA and $\overline{L}$ in period 0. Therefore, $L_0 = \overline{L} = \left(\frac{\gamma}{\rho}\right)^{\frac{1}{\alpha}} AX$ and $Y_0 = (\delta AX)^{\alpha} L_0^{1-\alpha} = \delta^{\alpha}\overline{Y}$, so that

$$y_0 = \frac{Y_0}{L_0} = \delta^{\alpha}\overline{y}$$

Therefore, $n_0 = \frac{\gamma}{\rho} y_0 = \delta^{\alpha}$, and $L_1 = n_0 L_0 = \delta^{\alpha}\overline{L}$.





7 – Ashraf & Galor [2011] write in their article:

> *"The Malthusian theory generates the following testable predictions:*
>
> *(i) Within a country, an increase in productivity would lead in the long run to a larger population, without altering the long-run level of income per capita.*
>
> *(ii) Across countries, those characterized by superior land productivity or a superior level of technology would have, all else equal, higher population densities in the long run, but their standards of living would not reflect the degree of their technological advancement"*

Prove that in the model we have solved, those testable predictions are correct.
Recall that

$$\overline{L} = \left(\frac{\gamma}{\rho}\right)^{\frac{1}{\alpha}} AX,$$

$$\overline{P}_d = \frac{\overline{L}}{X} = \left(\frac{\gamma}{\rho}\right)^{\frac{1}{\alpha}} A,$$

$$\overline{y} = \frac{\rho}{\gamma}.$$

Prediction (i) implies $\frac{\partial \overline{L}}{\partial A} > 0$ and $\frac{\partial \overline{y}}{\partial A} = 0$. Both are true in the model.
Prediction (ii) implies $\frac{\partial \overline{P}_d}{\partial A} > 0$ and $\frac{\partial \overline{y}}{\partial A} = 0$. Both are true in the model.

8 – Ashraf & Galor [2011] are proposing in their article a test of their model predictions (*Note: CE stands for "Common Era" and is a synonym for AD.*).

> *"The empirical examination of the central hypothesis of the Malthusian theory exploits exogenous sources of cross-country variation in land productivity and tech nological levels to examine their hypothesized differential effects on population density and income per capita during the time period 1-1500 CE"*

The authors first show that the Neolithic Revolution triggered a cumulative process of economic development, conferring a developmental head start to societies that experienced the agricultural transition earlier. Therefore, cross-country variation in the timing of the Neolithic Revolution can be used as a proxy for the variation in the level of technological advancement across countries during the agricultural stage of development. In simple words, the variable "Years since Neolithic transition" can be used as a measure of technology.

> *"Formally, the baseline specification adopted to test the Malthusian predictions regarding the effects of land productivity and the level of technology advancement on population density and income per capita are*

$$\begin{aligned}
\ln P_{i,t} &= \alpha_{0,t} + \alpha_{1,t} \ln T_i + \alpha_{2,t} \ln X_i + \alpha'_{3,t}\boldsymbol{\Gamma_i} + \alpha'_{4,t}\boldsymbol{D_i} + \delta_{i,t} \\
\ln y_{i,t} &= \beta_{0,t} + \beta_{1,t} \ln T_i + \beta_{2,t} \ln X_i + \beta'_{3,t}\boldsymbol{\Gamma_i} + \beta'_{4,t}\boldsymbol{D_i} + \varepsilon_{i,t}
\end{aligned}$$

where $P_{i,t}$ is the population density of country i in year t; $y_{i,t}$ is country i's income per capita in year t; T_i is the number of years elapsed since the onset of agriculture in country i; X_i is a measure of land productivity for country i, based on the percentage of arable land and an index of agricultural suitability; $\mathbf{\Gamma_i}$ is a vector of geographical controls for country i, including absolute latitude and variables gauging access to waterways; $\boldsymbol{D_i}$ is a vector of continental dummies; and, $\delta_{i,t}$ and $\varepsilon_{i,t}$ are country-specific disturbance terms for population density and income per capita, respectively, in year t.

Table 1 below reproduces the result of the estimation of those two equations on a sample of 31 countries (I am not reporting standard errors, only significance levels). The dummy variables used are Log Absolute Latitude, Mean Distance to Nearest Coast or River, Percentage of Land within 100 km of Coast or River and the continent to which the country belongs.

Table 1: Effects on Income per Capita versus Population Density

	Dependent Variable is:					
	Log Income per Capita in:			Log Population Density in:		
	1500 CE	1000 CE	1 CE	1500 CE	1000 CE	1 CE
	(1)	(2)	(3)	(4)	(5)	(6)
Years since Neolithic transition	0.159	0.073	0.109	1.337**	0.832**	1.006**
Log Land Productivity	0.041	-0.021	-0.001	0.584***	0.364***	0.681**
Log Absolute Latitude	-0.041	0.060	-0.175	0.050	-2.140**	-2.163**
Mean Distance to Nearest Coast or River	0.215	-0.111	0.043	-0.429	-0.237	0.118
Percentage of Land within 100 km of Coast or River	0.124	-0.150	0.042	1.855**	1.326**	0.228
Continent Dummies	Yes	Yes	Yes	Yes	Yes	Yes
Observations	31	26	29	31	26	29
R-squared	0.66	0.68	0.33	0.88	0.95	0.89

*Notes: (i) log land productivity is the first principal component of the log of the percentage of arable land and the log of an agricultural suitability index; (ii) a single continent dummy is used to represent the Americas, which is natural given the historical period examined; (iii) regressions (2)-(3) and (5)-(6) do not employ the Oceania dummy due to a single observation for this continent in the corresponding regression samples, restricted by the availability of income per capita data; (iv) *** denotes statistical significance at the 1 percent level, ** at the 5 percent level, and * at the 10 percent level, all for two-sided hypothesis tests. The absence of stars means that the coefficient is not significant at a 10 percent level.*

Interpret and comment those results through the lens of the model.

Consistent with the predictions of the Malthusian theory, the results demonstrate highly statistically significant positive effects of land productivity and the number of years elapsed since the Neolithic Revolution on population density in the years 1500 CE, 1000 CE, and 1 CE. The effects of these explanatory channels on income per capita in the corresponding periods, however, are not significantly differentfrom zero, a result that fully complies with Malthusian priors. These results are shown to be robust to controls for other geographical factors, including absolute latitude, access to waterways, distance to the nearest technological frontier, the percentage of land in tropical versus temperate climatic zones, and small island and landlocked dummies, all of which may have had an impact on aggregate productivity either directly, by affecting the productivity of land, or indirectly by affecting trade and the diffusion of technologies. Moreover, as foreshadowed by the initial findings in Table 1 of the Ashraf & Galor paper, the results are qualitatively unaffected when the index of technological sophistication, rather than the number of years elapsed since the Neolithic Revolution, is employed as a proxy for the level of aggregate productivity