University College London, 2019-2020
Econ 0107 – Macroeconomics I – Ralph Luetticke & Franck Portier

Midterm Exam

This is a 2 hours exam. No documents allowed. 100 points in total, 25 points for each problem.

I – The Welfare Cost of Business Cycles

Let utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^t u(c_t)$$

where instantaneous utility takes this specification $u(c_t) = \log(c_t)$.
The consumption process is

$$c_t = c_{t-1}^{\alpha}\epsilon_t exp(\mu)$$

where $\mu = -\frac{\sigma_\epsilon^2(1-\alpha)}{2(1-\alpha^2)}$ and $\log\epsilon_t \sim iid - N(0,\sigma_\epsilon^2)$. So that the logarithm of consumption follows an AR(1) with parameter α:

$$\log c_t = \mu + \alpha\log c_{t-1} + \log\epsilon_t$$

1 – In the long run, $\log c_t$ converges (in distribution) to $N(\frac{\mu}{1-\alpha}, \frac{\sigma_\epsilon^2}{1-\alpha^2})$, which we call the invariant distribution. Show that the normalization proposed is such that the mean consumption is equal to one.
Hint: Let Z be a standard normal variable, and let μ_Z and $\sigma_Z > 0$ be two real numbers. Then, the distribution of the random variable $X = e^{\mu_Z+\sigma_Z Z}$ is called the log-normal distribution with parameters μ_Z and σ_Z. The mean of a lognormal distribution is given by $\exp(\mu_Z + \sigma_Z^2/2)$.

Using the definition of the mean of lognormal and plugging in for μ yields

$$E(c) = E[\exp(\log c)] = \exp(\frac{\mu}{1-\alpha} + \frac{\sigma_\epsilon^2}{2(1-\alpha^2)}) = 1$$

2 – Assuming that consumption at time 0 (c_0) is known, find the value function

$$V(\lambda,\sigma_\epsilon^2) = E_0\sum_{t=0}^{\infty}\beta^t u(c_t(1+\lambda)),$$

where λ is a positive constant.
Hint: You can write the problem recursively as:

$$V(c;\lambda,\sigma_\epsilon^2) = u(c(1+\lambda)) + \beta EV(c';\lambda,\sigma_\epsilon^2)$$

and use a guess-and-verify approach: $V(c;\lambda,\sigma_\epsilon^2) = A\log c + B$

Using the guess one can write:

$$A\log c + B = \log[c(1+\lambda)] + \beta E(A\log[c^\alpha\epsilon exp(\mu)] + B$$

$$A\log c + B = \log c + \log(1+\lambda) + \beta A\alpha\log c + \beta AE\log\epsilon + \beta A\mu + \beta B,$$

where $E\log\epsilon = 0$ so that matching coefficients yields

$$A = \frac{1}{(1-\alpha\beta)}$$

$$B = \frac{\log(1+\lambda)}{(1-\beta)} + \frac{\beta\mu}{(1-\beta)(1-\alpha\beta)}$$

so that we have

$$V(c;\lambda,\sigma_\epsilon^2) = \frac{\log c}{(1-\alpha\beta)} + \frac{1}{(1-\beta)}\log(1+\lambda) + \frac{\beta\mu}{(1-\beta)(1-\alpha\beta)}$$

3 − Compute the value of λ as a function of α, σ_ϵ^2 and γ such that

$$V(c;\lambda,\sigma_\epsilon^2) = V(c;0,0).$$

What does λ represent? Determine how λ depends on α, σ_ϵ^2 and γ and interpret you results. Compare to the i.i.d. case, i.e. $\alpha = 0$. Hint: try to use $\log(1+x) \approx x$.

Setting $V(c;\lambda,\sigma_\epsilon^2) = V(c;0,0)$, we are looking for λ such that

$$\log(1+\lambda) = -\frac{\beta\mu}{(1-\alpha\beta)}$$

Plugging in for μ

$$\log(1+\lambda) = \frac{\beta(1-\alpha)\sigma_\epsilon^2}{2(1-\alpha^2)(1-\alpha\beta)}$$

or

$$\lambda \approx \frac{\beta\sigma_\epsilon^2}{2(1+\alpha)(1-\alpha\beta)}$$

When $\alpha = 0$ we have that

$$\lambda \approx \frac{\beta\sigma_\epsilon^2}{2}$$

4 − What happens when $\alpha \to 1$? Is it sensible to think at λ as the "welfare cost of business cycles"?

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Plugging in for μ

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or

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II – McCall Model

Consider a worker who draws every period a job offer to work forever at wage w. Successive offers are independently and identically distributed drawings from a distribution $F_i(w)$, $i = 1, 2$. Assume that F_1 has been obtained from F_2 by a mean-preserving spread. The worker's objective is to maximize

$$E\sum_{t=0}^{T}\beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ if the worker has accepted employment at wage w and is zero otherwise. Assume that both distributions, F_1 and F_2, share a common upper bound, B.

1 – Write down the Bellman equation at time t. Argue that the optimal policy is of the reservation wage form.

$$v_t^i = \max\left\{ w\frac{1-\beta^{T-t+1}}{1-\beta}, \beta\int_0^B v_{t+1}^i(w')dF_i(w')\right\}$$

where $w\frac{1-\beta^{T-t+1}}{1-\beta}$ is the value of working at wage w in periods $t, t+1, ..., T$, The first term is increasing in w while the second one is constant. It follows that, at time t, the optimal policy is of the reservation wage form.

2 – Show that the reservation wages of workers drawing from F_1 and F_2 coincide at $t = T$ and $t = T-1$.

$$v_T^i = \max\{0, w\} = w$$

Therefore,

$$\int_0^B v_T^i(w)dF_i(w) = \int_0^B wdF_i(w) = Ew, \quad i = 1, 2$$

The reservation wage at time T is zero: the worker accepts every offer. At time $(T-1)$, Bellman equation reads

$$v_{T-1}^i = \max\left\{ w(1+\beta), \beta\int_0^B v_T^i(w')dF_i(w')\right\} = \max\{ w(1+\beta), \beta Ew\}$$

It is then clear that the worker will accept the offer if $w(1+\beta) \geq \beta Ew$ and will reject it otherwise. Therefore the reservation wage w_{T-1} is $\beta Ew/(1+\beta)$. Because the expectation of w is the same no matter whether w is drawn from F_1 or F_2 , it follows that both types of workers have the same reservation wage.

3 – Now introduce unemployment compensation: the worker who is unemployed collects c dollars. Show that the result in **2** – no longer holds, that is, the reservation wage of the workers that sample from F_1 is higher than the one corresponding to workers that sample from F_2 for $t = T-1$.

The problem at time T now reads

$$v_T^i = \max\{c, w\}.$$

Then, $\bar{w}_1 = \bar{w}_2 = c$. However,

$$\int_0^B max\{w, c\}dF_1(w) \geq \int_0^B max\{w, c\}dF_2(w)$$

or $Ev_T^1 \geq Ev_T^2$. Hence, $\bar{w}_{T-1}^1 \geq \bar{w}_{T-1}^2$ as the reservation wage is equal to $\beta/(1+\beta)Ev_T^i$

III – Fiscal Policies in a Dynamic Economy

Consider the following setup. There is no uncertainty, and decision makers have perfect foresight. A representative household has preferences over nonnegative streams of a single consumption good $c_t t$ that are ordered by

$$\sum_{t=0}^{\infty}\beta^t u(c_t), \qquad \beta \in]0, 1[$$

where u is strictly increasing in c_t, twice continuously differentiable, and strictly concave. The households supplies inelastically one unit of labor.

The technology is

$$g_t + c_t + x_t \leq F(k_t, n_t)$$
$$k_{t+1} = (1-\delta)k_t + x_t$$

where $\delta \in]0, 1[$ is a depreciation rate, k_t is the stock of physical capital, x_t is gross investment, and $F(k, n)$ is a linearly homogeneous production function with positive and decreasing marginal products of capital and labor. We have the notation $F(k, 1) = f(k)$. g stands for government expenditure.

There is a competitive equilibrium with all trades occurring at time 0. The household owns capital, makes investment decisions, and rents capital and labor to a representative production firm. The representative firm uses capital and labor to produce goods with the production function $F(k_t, n_t)$. A price system is a triple of sequences $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ where q_t is the time 0 pretax price of one unit of investment or consumption at time t (x_t or c_t), r_t is the pretax price at time 0 that the household receives from the firm for renting capital at time t, and w_t is the pretax price at time 0 that the household receives for renting labor to the firm at time t.

The household faces the budget constraint:

$$\sum_{t=0}^{\infty}\Big(q_t(1+\tau_{ct})c_t + (1-\tau_{it})q_t\big[k_{t+1}-(1-\delta)k_t\big]\Big) \leq \sum_{t=0}^{\infty}\Big(r_t(1-\tau_{kt})k_t + w_t(1-\tau_{nt})n_t - q_t\tau_{ht}\Big)$$

where τ_{ct} is the consumption tax rate, τ_{it} the investment credit rate, τ_{kt} the capital earning tax rate, τ_{nt} the labor earning tax rate and τ_{ht} a lump-sum tax.

1 – Write the government budget constraint.

$$\sum_{t=0}^{\infty} q_t\Big(g_t + \tau_{it}(k_{t+1}-(1-\delta)k_t)\Big) \leq \sum_{t=0}^{\infty}\Big(q_t\tau_{ct}c_t + \tau_{kt}r_tk_t + \tau_{nt}w_tn_t + q_t\tau_{ht}\Big)$$

2 – Define a competitive equilibrium.

A competitive equilibrium is a budget feasible policy, a feasible allocation and a price system such that, given price system and government policy, *(i)* the allocation solves the household problem and *(ii)* the allocation solves the firm problem.

3 – Use the household budget constraint to derive a no-arbitrage condition, from which you will get a formula for the user cost of capital. Comment.

The household budget constraint writes:

$$\begin{aligned}\sum_{t=0}^{\infty} q_t(1+\tau_{ct})c_t \leq &\sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t - \sum_{t=0}^{\infty} q_t\tau_{ht}\\ &+\sum_{t=0}^{\infty}\Big(r_t(1-\tau_{kt}) + q_t(1-\tau_{it})(1-\delta)k_t - q_{t-1}(1-\tau_{it-1})\Big)k_t\\ &-\lim_{T\to\infty}(1-\tau_{iT})q_Tk_{T+1}\end{aligned}$$

By arbitrage,

$$q_t(1-\tau_{it}) = q_{t+1}(1-\tau_{it+1})(1-\delta) + r_{t+1}(1-\tau_{it+1}). \qquad (a)$$

If (a) does not hold for some t, then the household could obtain an infinite wealth by choosing k_t going to + or - ∞. She would then consume plus infinity, which cannot be an equilibrium with finite endowments.

4 – Write down the household maximization problem and derive first order conditions.

5 – Write down the firm maximization problem and derive first order conditions.

6 – Show that the equilibrium can be characterized by a second order nonlinear difference equation in k_t.

7 – Derive a formula for the steady state level of capital.

8 – Which taxes are distortionary at the steady state?

IV – Overlapping genearations

Consider an economy consisting of overlapping generations of two-period-lived agents. There is a constant population of N young agents born at each date $t \geq 1$. There is a single consumption good that is not storable. Each agent born in $t \geq 1$ is endowed with w_1 units of the consumption good when young and with w_2 units when old, where $0 < w_2 < w_1$. Each agent born at t$\geq$ 1 has identical preferences $\ln c_t^h(t) + \ln c_t^h(t+1)$, where $c_t^h(s)$ is time s consumption of agent h born at time t. In addition, at time 1, there are alive N old people who are endowed with $H(0)$ units of unbacked paper currency and who want to maximize their consumption of the time 1 good.

A government attempts to finance a constant level of government purchases $G(t) = G \geq 0$ for $t \geq 1$ by printing new base money. The government's budget constraint is

$$G = (H(t) - H(t-1))/p(t),$$

where $p(t)$ is the price level at t, and $H(t)$ is the stock of currency carried over from t to $(t+1)$ by agents born in t. Let $g = G/N$ be government purchases per young person. Assume that purchases $G(t)$ yield no utility to private agents.

1 – Denote $s_t^h(t)$ the savings of young agent h, and let $r(t)$ be the rate of return of savings between t and $(t+1)$. Write down the maximization problem of agent h, and derive. saving function $f^h(1+r(t))$.

2 – Let's assume that savings can be done using currency or by issuing/subscribing privately issued bonds. Explain why the following equation holds in equilibrium when fiat currency (money) is valued:

$$1 + r(t) = \frac{p(t)}{p(t+1)}.$$

3 – Define a competitive equilibrium with valued fiat currency.

4 – Find a stationary equilibrium with valued fiat currency.

5 – Prove that for small enough g, there exist two stationary equilibria with valued fiat currency. (*You can do that graphically by using the function f to draw seigniorage revenues as a function of* $(1+r)$).

SOLUTION TO ECON 0107 EXAM – FIRST TERM

Problem III – FISCAL POLICIES IN A DYNAMIC ECONOMY

1– Gvt BC :

$$\sum_{t=0}^{\infty} q_t \left(g_t + \tau_{it} q_t \left(k_{t+1} - (1-\delta) k_t \right) \right) \leq \sum_{t=0}^{\infty} \left(q_t \tau_{ct} c_t + \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + q_t \tau_{ht} \right)$$

2– Competitive equilibrium :

- a budget feasible policy
- a feasible allocation
- a price system

such that, given price system and gvt. policy,

- the allocation solves the household pb
- ——————— firm —

3 – The household budget constraint writes

$$\sum_{t=0}^{\infty} q_t (1+\tau_{c_t}) c_t \leq \sum_{t=0}^{\infty} (1-\tau_{n_t}) w_t n_t - \sum_{t=0}^{\infty} q_t \tau_{h_t} + \left(r_0 (1-\tau_{k_0}) + (1-\tau_{i0}) q_0 (1-\delta) \right) k_0$$

$$+ \sum_{t=1}^{\infty} \left(r_t (1-\tau_{k_t}) + q_t (1-\tau_{i_t})(1-\delta) - q_{t-1} (1-\tau_{i_{t-1}}) \right) k_t - \lim_{T \to \infty} (1-\tau_{i_T}) q_T k_{T+1}$$

By arbitrage $\quad q_t (1-\tau_{i_t}) = q_{t+1} (1-\tau_{i_{t+1}})(1-\delta) + r_{t+1} (1-\tau_{k_{t+1}}) \quad (a)$

(if not, choosing $k_t \to \pm \infty$ would allow to ensure $+\infty$)

user cost of capital $\quad r_{t+1} = \left(\frac{1}{1-\tau_{k_{t+1}}} \right) \left(q_t (1-\tau_{i_t}) - q_{t+1} (1-\tau_{i_{t+1}}) + \delta q_{t+1} (1-\tau_{i_{t+1}}) \right)$

The user cost of capital takes into account the rate of taxation of capital earnings, the capital gain or loss from t to $t+1$ and the investment-credit-adjusted-depreciation cost.

4 - Household problem : $\max_{\{c_t\}} \sum \beta^t u(c_t) + \mu \cdot$ Intertemp. Budget Constraint

FOC $\quad \beta^t u'_t = \mu\, q_t(1+\tau_{ct}) \quad$ (b)

5 - Firm $\quad \max_{\{n_t, k_t\}} F(k_t, n_t) - w_t n_t - r_t k_t$

FOC : $\quad r_t = q_t F_{kt} = q_t f'(k_t) \quad$ (c)

$w_t = q_t F_{nt}$

6 -

take (a), replace q by the expression in (b) and r_{t+1} by the expression in (c) + use $c_t = f(k_t) - k_{t+1} + (1-\delta) k_t$

$$u'\left(f(k_t) - k_{t+1} + (1-\delta)k_t\right) = \beta\, u'\left(f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}\right) \times \left[\frac{1-\tau_{ct+1}}{1-\tau_{ct}}(1-\delta) + \frac{1-\tau_{kt+1}}{1-\tau_{ct}} f'(k_{t+1}) \right] \quad (*)$$

7- At the steady state, $1 = \beta\left(1-\delta + \left(\frac{1-\tau_k}{1-\tau_i}\right) f'(k)\right)$

8- τ_c, τ_n, τ_h are non distortionary (τ_h, τ_n for obvious reasons, τ_c only at the SS)

τ_i, τ_k are distortionary

PROBLEM IV : OVERLAPPING GENERATIONS

1- $\max \ \ln c_t^h(t) + \ln c_t^h(t+1) + \lambda\left(w_1 + (1+r(t))^{-1} w_2 - c_t^h(t) - (1+r(t))^{-1} c_t^h(t+1)\right)$

FOC : $1/c_t^h(t) = \lambda$, $1/c_t^h(t+1) = (1+r(t))^{-1}\lambda$

$\Rightarrow \quad c_t^h(t) = c_t^h(t+1)/(1+r(t))$

plug in the BC: $w_1 + (1+r(t))^{-1} w_2 = 2 c_t^h(t) \Rightarrow s_t^h = w_1 - c_t^h(t) = \frac{1}{2}\left(w_1 - \frac{w_2}{1+r(t)}\right)$

$= f^h(1+r(t))$

Remark: $f^h(1+r(t)) = f(1+r(t))$ [does not depend on h]

2 – $p(t)/p(t+1)$ is the return on money

$1+r(t)$ is the return on bonds

No uncertainty $\Rightarrow$ by arbitrage, all assets that are positively valued have the same return

$\Rightarrow p(t)/p(t+1) = 1+r(t)$.

3 – · All agents behave the same $\Rightarrow$ drop the h index.

Def: An equilibrium with valued fiat currency is $\{\{r(t), \{p(t)\}, \{H(t)\}\}$ and allocations $\{c_t^h(t)\}$ for all $t \in N$

such that

- $1+r(t) = p(t)/p(t+1)$ (both assets are held) (a)
- $G = (H(t) - H(t-1))/p_t$ (Gvt budget constraint) (b)
- $\sum_h f^h(1+r(t)) = N f(1+r(t)) = H(t)/p(t) \quad \forall t \geq 1$ [utility max + market clearing] (c)
- $c_t^h(t) = w_1 - f(1+r(t))$, (d) $c_t^h(t+1) = w_2 + (1+r(t)) f(1+r(t))$ (e)

4- Real variables will be constant, nominal ones will grow.

Using the equations defining the equilibrium;

(c) $\Rightarrow$ $f(1+r) = \frac{M(t)}{N\, p(t)}$ $\Rightarrow$ $\frac{M(t)}{p(t)}$ is constant

(b) $\Rightarrow$ $g = \frac{M(t)}{N\, p(t)} - \frac{p(t-1)}{p(t)} \times \frac{M(t-1)}{N\, p(t-1)}$

using (c): $\underbrace{\frac{M(t)}{N\, p(t)}}_{f(1+r)}$, $\underbrace{\frac{p(t-1)}{p(t)}}_{(1+r)}$, $\underbrace{\frac{M(t-1)}{N\, p(t-1)}}_{f(1+r)}$

$\Rightarrow$ eq. is found by solving the equation $g = f(1+r) - (1+r) f(1+r)$

or $g = h(r)$

with $h(r) = -\frac{r}{2}\left(w_1 - \frac{w_2}{1+r}\right)$

5– check that $h(0) = h\left(\frac{w_2}{w_1} - 1\right) = 0$, $\frac{w_2}{w_1} - 1 < 0$

$$h'(r) = -\frac{1}{2}\left(w_1 - \frac{w_2}{1+r} \right) + 2 \frac{w_2}{(1+r)^2} \Big)$$

$$h'(0) < 0, \quad h'\left(\frac{w_2}{w_1} - 1\right) > 0$$

↳