University College London, 2020-2021
Econ 0107 – Macroeconomics I – Ralph Luetticke & Franck Portier

Midterm Coursework Assessment

Each part carries 25% of the total mark

By submitting this assessment, I pledge my honour that I have not violated UCL's Assessment Regulations which are detailed in the UCL academic manual (chapter 6, section 9 on student academic misconduct procedure), which include (but are not limited to) plagiarism, self-plagiarism, unauthorised collaboration between students, sharing my assessment with another student or third party, access another student's assessment, falsification, contract cheating, and falsification of extenuating circumstances.

I – The End of the World in an OLG Model

This problem explores the consequences of the world having an end (possibly stochastically) in an OLG model. Notations will be, as much as possible, those of the course. Agents are indexed by $i = 0, 1, ..., +\infty$, where i is the period of birth. The economy starts in period 1. Each generation lives for two periods, and preferences are

$$U^i(c^i) = \log(c_i^i) + \log(c_{i+1}^i).$$

Utility is $U^0(c^0) = \log(c_1^0)$ for the old of period 0 and $\log(c_i^i)$ for the young in the last period of the world. Endowments are $(y_i^i, y_{i+1}^i) = (1-\varepsilon, \varepsilon)$ and $y_t^i = 0$ if $t \neq i, i+1$, with $\varepsilon \in]0, 1/2[$.

The part that is different from the course is that, in every period i, Nature is flipping a coin and the world will end at the end of the next period $(i+1)$ with probability π_i and will survive the end of the next period with probability $1 - \pi_i$.

In period i, the economy can be in three different states:

- $s_i = E$ if the world is ending at the end of $i+1$;
- $s_i = \overline{E}$ if the world does not end at the end of $i+1$;
- $s_i = \varnothing$ if period i is the last period of the world (meaning that $s_{i-1} = E$).

We always assume that $\pi_1 = 0$.

★

Assume first that $\pi_1 = 0$ and $\pi_2 = 1$. In words, the economy is deterministic, and ends for sure in period 3.

1 – Assume time-0 trading and no money. Write budget constraints of agents of generations 0 to 3 (use prices q_i^0). Compute their offer curves. Show that autarky is an equilibrium and that the perfect smoothing allocation

$$\left(c_1^0, \left\{c_i^i, c_{i+1}^i\right\}_{i=1,2}, c_3^3\right) = \left(\frac{1}{2}, \left\{\frac{1}{2}, \frac{1}{2}\right\}, \frac{1}{2}\right)$$

is *not* and equilibrium. Discuss.

2 – Assume now sequential trading. A quantity of money M is distributed to the old of period 1. Denote p_i the price of one unit of good in terms of the currency. Show that there cannot be any monetary equilibrium.

★

Now assume that the world never ends, –*i.e.* $\pi_i = 0 \; \forall i \geq 1$.

3 – What are the two stationary equilibria with time-0 trading (and no money)?

4 – What is the unique stationary monetary equilibrium with sequential trading?

★

Enough for the warm-up. Assume now that $\pi_i = \pi \in]0,1[\forall i \geq 2$. For an active period i (by active, we mean not after the end of the world), denote $x_i(s_i)$ any variable of period i, with $s_i \in \{E.\overline{E}, \varnothing\}$.

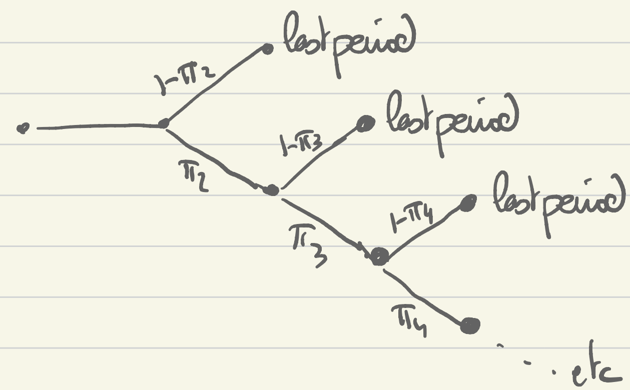
5 – Assume time-0 trading and no money. Write the maximization problem of generation $i > 1$ (expected utility and intertemporal budget constraint). Keep in mind that prices depend on the state of the world. Derive first order conditions. Can autarky be an equilibrium? Can perfect smoothing be an equilibrium? Discuss.

6 – Assume now sequential trading and that the old of generation is endowed with M units of money. Can perfect smoothing be an equilibrium? Find a condition on π for the existence of a stationary monetary equilibrium. What are the equilibrium quantities at this equilibrium. Discuss.

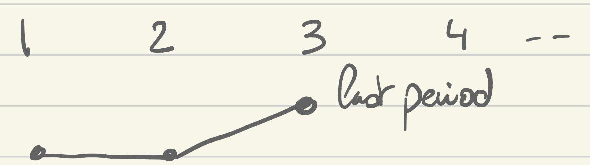
I - THE END OF THE WORLD IN AN OLG MODEL -

Model is: period 1 2 3 4 5 ....

(with the restriction $\pi_1 = 0$)



1 - We look at the following deterministic path:



Normalize $q_1^0 = 1$

Budget constraints are

gen 0 : $c_1^0 \leq \varepsilon \quad (1)$

gen 1 : $c_1^1 + q_2^0 c_2^1 = 1-\varepsilon + q_2^0 \varepsilon$

gen 2 $\quad q_2^0 c_2^2 + q_3^0 c_3^2 = q_2^0 (1-\varepsilon) + q_3^0 \varepsilon$

gen 3 $\quad c_3^3 \leq 1-\varepsilon \quad (2)$

Define $\alpha_i = \frac{q_{i+1}^0}{q_i^0}$, FOC for generation i is $\frac{c_i^i}{c_{i+1}^i} = \alpha_i \quad (3)$

and BC is $c_i^i + \alpha_i c_{i+1}^i = 1-\varepsilon + \alpha_i \varepsilon \quad (4)$

(3) $\Rightarrow c_i^i = \alpha_i c_{i+1}^i \quad \rightarrow$ plug in (4) : $\left(2 - \frac{\varepsilon}{c_{i+1}^i}\right) c_i^i = (1-\varepsilon) \quad (5)$ ← OFFER CURVE

Def: An equilibrium is $\left(c_1^0, \{c_i^i, c_{i+1}^i\}_{i=1,2}, c_3^3\right)$ s.t.

- (1), (2), (5) are satisfied
- alloc. are feasible: $c_i^{i-1} + c_i^i \leq 1$

• Can autarky be an equilibrium?

- it is feasible
- Are there prices for which (1), (2) (5) are satisfied?

yes, if $q_1 = \frac{1-\varepsilon}{\varepsilon}$, $q_2 = \frac{1-\varepsilon}{\varepsilon}$, $R_1 = R_2 = \frac{1}{q} = \frac{\varepsilon}{1-\varepsilon} < 1$

↳ low interest rate eq.

• Can perfect smoothing be an equilibrium?

For gen 1 and 2: we need $q_1 = q_2 = \frac{1/2}{1/2} = 1 \Rightarrow R_1 = R_2 = 1$

but for generation 3: $c_3^3 = \frac{1}{2}$ is not an optimal choice

$c_3^3 = 1-\varepsilon$ is the optimal choice

$\Rightarrow$ Not an equilibrium.

$\rightarrow$ the last period kills the perfect smoothing equilibrium.

2 – With money, generation i problem is: $\max \log c_i^i + \log c_{i+1}^i$

$$\text{s.t.} \quad c_i^i + \frac{m_i^i}{P_i} \leq 1-\varepsilon$$

$$c_{i+1}^i \leq \frac{m_i^i}{P_{i+1}} + \varepsilon$$

$$m_i^i \geq 0$$

Take the last period $(t=3)$

The young of the last period BC is $c_3^3 \leq 1-\varepsilon - \frac{m_3^3}{P_3}$

and their optimal choice is $m_3^3 = 0$ (No reason to save as they will not live the next period)

Therefore, given that in equilibrium $\frac{m_2^2}{P_3} = \frac{m_3^3}{P_3} = 0$

such that BC of gen 2 old is $c_3^2 = \frac{m_2^2}{P_3} + \varepsilon = \varepsilon$

given that $\frac{m_2^2}{P_3} = 0$, it is optimal for gen 2 to choose $m_2^2 = 0$

The same reasoning applies to generation 1 : $m_1^1 = 0$

Therefore, money market equilibrium in period 1 implies $\frac{M}{p_1} = \frac{m_1^1}{p_1} = 0$

$\Rightarrow p_1 = +\infty$ (money has no value)

$\longrightarrow$ the only equilibrium is therefore autarky.

*

We now move to a model with infinite horizon. That is exactly the one we have seen in class.

3- Time-0 Trading

$$\begin{cases} \text{Max } \log c_i^i + \log c_{i+1}^i \\ \text{s.t. } q_i^0 c_i^i + q_{i+1}^0 c_{i+1}^i \leq q_i^0 (1-\varepsilon) + q_{i+1}^0 \varepsilon \end{cases}$$

∞

Let $\alpha_i = \frac{q_{i+1}^0}{q_i^0}$, offer curve is given by $c_i^i + \alpha_i c_{i+1}^i \leq (1-\varepsilon) + \alpha_i \varepsilon$

$$\frac{c_i^i}{c_{i+1}^i} = \alpha_i$$

$$\Rightarrow \alpha_i = \frac{1}{\varepsilon} - \frac{\frac{1}{\varepsilon} - 1}{\alpha_{i-1}} \quad (6)$$

stationary solutions to (G) are

- $\alpha = 1 \Rightarrow c_i^i = c_{i+1}^i = \frac{1}{2}$ and $c_1^0 = \frac{1}{2}$ is feasible

  because $\frac{1}{2} + \frac{1}{2} = 1 = \varepsilon + 1 - \varepsilon$

- $\alpha = \frac{1-\varepsilon}{\varepsilon} \Rightarrow c_i^i = 1-\varepsilon, \; c_{i+1}^i = \varepsilon, \; c_1^0 = \varepsilon$

  = autarky.

$\alpha = 1$ is the high interest rate stationary equilibrium and $\alpha = \frac{1-\varepsilon}{\varepsilon}$ is the low one.

L_1. with sequential trading and money,

generation i $\quad \max \log c_i^i + \log c_{i+1}^i$

$$\text{s.t.} \quad c_i^i + \frac{m_i^i}{p_i} \leq 1 - \varepsilon$$

$$c_{i+1}^i \leq \frac{m_i^i}{p_{i+1}} + \varepsilon$$

$$m_i^i \geq 0$$

$\rightarrow$ saving function $\quad y_i^i - c_i^i = s(\alpha_i)$

$$= \frac{1}{2}(1 - \varepsilon - \alpha_i \varepsilon)$$

with $\alpha_i = \frac{p_{i+1}}{p_i}$

equilibrium :

$$\begin{cases} \dfrac{\Pi}{p_i} = s(d_i) \\ d_i = \dfrac{p_{i+1}}{p_i} \\ s(d_i) = \frac{1}{2}\left(1-\varepsilon - d_i \varepsilon\right) \end{cases}$$

$$2\Pi = p_i\left(1-\varepsilon - p_{i+1}\varepsilon\right)$$

$$p_i = \frac{2\Pi}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon} p_{i+1}$$

$$\Leftrightarrow \frac{\Pi}{p_i} = \frac{1}{2}\left(1-\varepsilon - \frac{p_{i+1}}{p_i}\varepsilon\right) \Leftrightarrow p_i = \frac{2\Pi}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon} p_{i+1} = \text{(solve forward)}$$

The only stationary eq. is $p = 2\Pi$, $d = 1$

and then $s = \frac{1-2\varepsilon}{2}$, $c_i^i = 1-\varepsilon - \frac{1-2\varepsilon}{2} = 1 - \frac{1}{2} - \varepsilon + \varepsilon = \frac{1}{2} \quad \forall i$

$c_{i+1}^i = \varepsilon + d\left(\frac{1-2\varepsilon}{2}\right) = \varepsilon + \frac{1}{2} - \varepsilon = \frac{1}{2} \quad \forall i$

$\hookrightarrow$ the stat. eq. is the perfect smoothing one.

☆

Now $\pi_t = \pi \in]0,1[$

5- Time 0 Trading

in period i, if the economy is active (meaning that the world has not ended before), 2 states are possible

- $s_i = E$: the end of the world is next period
- $s_i = \bar{E}$: the end of the world is not for the next period
- $s_i = \phi$: the end of the world is for the end of this period.

As of time 0, generation $i \geq 1$ maximizes

$$(1-\pi)^{i-1}\left[\pi\left(\log c_i^i(E)+\log c_{i+1}^i(\phi)\right)+(1-\pi)\left(\log c_i^i(\bar{E})+(1-\pi)\log c_{i+1}^i(\bar{E})+\pi\log c_{i+1}^i(E)\right)\right]+(1-\pi)^{i-2}\,\pi\, c_i^i(\phi)$$

- $(1-\pi)^{i-1}$ ↑ the world has not ended before
- π ↑ the world ends next period
- $(1-\pi)$ ↑ the world does not end next period
- $(1-\pi)\log c_{i+1}^i(\bar{E})$ ↑ the world does not end in $i+2$
- $\pi \log c_{i+1}^i(E)$ ↑ the world ends in $i+2$
- $(1-\pi)^{i-2}\pi\, c_i^i(\phi)$ ↑ the world ends this period

and generation i budget constraint is

$$q_i(\phi)\, c_i^i(\phi) + q_i(E)\, c_i^i(E) + q_{i+1}\, c_{i+1}^i(\phi) + q_i(\bar{E})\, c_i^i(\bar{E}) + q_{i+1}(\bar{E})\, c_{i+1}^i(\bar{E}) + q_{i+1}(E)\, c_{i+1}^i(E)$$

$$\leq \big(q_i(E) + q_i(\bar{E}) + q_i(\phi)\big)(1-\varepsilon) + \big(q_{i+1}(E) + q_{i+1}(\bar{E}) + q_{i+1}(\phi)\big)\,\varepsilon \qquad (\lambda^i)$$

Note: As there is no trade after the end of the world and no utility, we write $q_{i+j}(s^j / s_i = \phi) = 0 \ \forall j \geq 1$

FOC are

$$(1-\pi)^{i-2}\, \pi \, \frac{1}{c_i^i(\phi)} = \lambda^i q_i(\phi) \quad (1) \qquad (1-\pi)^{i-1}\, \pi \, \frac{1}{c_{i+1}^i(\phi)} = \lambda^i q_{i+1}(\phi) \quad (4)$$

$$(1-\pi)^{i-1}\, \pi \, \frac{1}{c_i^i(E)} = \lambda^i q_i(E) \quad (2) \qquad (1-\pi)^{i}\, \pi \, \frac{1}{c_{i+1}^i(E)} = \lambda^i q_{i+1}(E) \quad (5)$$

$$(1-\pi)^{i-1} \times (1-\pi) \, \frac{1}{c_i^i(\bar{E})} = \lambda^i q_i(\bar{E}) \quad (3) \qquad (1-\pi)^{i}(1-\pi) \, \frac{1}{c_{i+1}^i(\bar{E})} = \lambda^i q_{i+1}(\bar{E}) \quad (6)$$

* Can $\left(\frac{1}{2},\frac{1}{2}\right)$ $\forall i \geq 1$ and $e_1^0 = \varepsilon$ be an equilibrium?

- it is feasible.

- prices will have to be $\frac{(4)}{(1)}$: $\frac{q_{i+1}(\emptyset)}{q_i(\emptyset)} = 1-\pi$ $\quad \frac{(5)}{(2)}$ $\frac{q_{i+1}(E)}{q_i(E)} = 1-\pi$ $\quad \frac{(6)}{(3)}$: $\frac{q_{i+1}(\bar{E})}{q_i(\bar{E})} = 1-\pi$

Normalizing, say, $q_1(\emptyset)=1$, we can find all the $q_i(s_i)$.

⇒ for those prices, perfect smoothing will be an equilibrium.

Note that the real interest factor is not 1 but $1-\pi<1$

↳ The risk of the world ending plays like a discount factor.

* Can $(1-\varepsilon, \varepsilon)$ be an equilibrium?

* it is feasible

* it is an eq. if prices are such that $\frac{q_{i+1}(s_i)}{q_i(s_i)} = \frac{\varepsilon}{1-\varepsilon} \times (1-\pi)$ $\forall s_i \in \{E, E, \emptyset\}$

6- Sequential trading with money.

- If $s_i = E$, then money does not have a finite price and equilibrium must be:

  $c_i^i(E) = 1-\varepsilon$ $\quad c_i^{i-1}(E) = \varepsilon$ $\quad$ and $\quad c_{i+1}^i(\emptyset) = \varepsilon$ $\quad c_{i+1}^{i+1}(\emptyset) = 1-\varepsilon$

- If $s_i = \emptyset$ $\quad c_i^{i-1}(\phi) = \varepsilon$, $\quad c_i^i(\emptyset) = 1-\varepsilon$

• if $s_i = \bar{E}$, the problem of generation i is (assuming a monetary equilibrium)

$$\max \; \log c_i^i(\bar{E}) + (1-\pi)\log c_{i+1}^i(\bar{E}) + \pi \log \underbrace{c_{i+1}^i(E)}_{\varepsilon \text{ as shown just before}}$$

$$\text{s.t. } c_i^i(\bar{E}) + \frac{P_{i+1}(\bar{E})}{P_i(\bar{E})} c_{i+1}^i(\bar{E}) \leq 1-\varepsilon + \frac{P_{i+1}(\bar{E})}{P_i(\bar{E})} \times \varepsilon \quad (\lambda^i)$$

$$\underline{\text{FOC}} \Rightarrow \frac{P_{i+1}(\bar{E})}{P_i(\bar{E})} = (1-\pi)\frac{c_i^i(\bar{E})}{c_{i+1}^i(\bar{E})} \quad (*)$$

→ Can we find a monetary equilibrium with constant price and constant $c_y(\bar{E})$ and $c_o(\bar{E})$? (as long as the end of the world is not drawn).

$p(\bar{E})$ = constant $\Rightarrow$ from (*) : $(1-\pi)c_o(\bar{E}) = c_y(\bar{E})$

$\rightarrow$ using the resource constraint $c_o(\bar{E}) + c_y(\bar{E}) = 1$

$\Rightarrow$ $c_y = \frac{1}{2-\pi}$, $c_o = \frac{1-\pi}{2-\pi}$

. We need to check that this is feasible with positive savings (so that $m^i \geq 0$)

$$\frac{m_i(\bar{E})}{p_i(\bar{E})} \geq 0 \iff c_y < 1-\varepsilon \iff \frac{1}{2-\pi} < 1-\varepsilon \iff \boxed{\pi < \frac{1-2\varepsilon}{1-\varepsilon}}$$

$\hookrightarrow$ * if the probability of the end of the world is not too large $\left(\pi < \frac{1-2\varepsilon}{1-\varepsilon}\right)$, then one can sustain a monetary equilibrium, as long as the end of the world is not drawn.

$\rightarrow$ there is more smoothing than in autarky $\Rightarrow$ higher welfare.

II – Business Cycles Movements at a Temporary Equilibrium

Data show that consumption, investment and labour all move in the same direction in the business cycle. This is what we call "business cycle movements". We are here interested in how the economy responds to a news about a future change in the economy, that does not affect current preferences nor technology. Can this create a business cycle movement?

The model we consider is a perfect competition complete market growth model. Let's assume the economy is deterministic. Preferences are

$$\sum_{t=0}^{\infty} \beta^t \Big(U(c_t) - V(L_t) \Big)$$

and technology is

$$Y_t = F(K_t, L_t)$$

and

$$K_{t+1} = (1-\delta)K_t + I_t.$$

It is assumed that $V_L > 0$, $V_{LL} > 0$, $U_C > 0$, $U_{CC} < 0$, $F_K > 0$, $F_{KK} < 0$, $F_L > 0$, $F_{LL} < 0$ and $F_{KL} > 0$. We also assume that F is concave. K_0 is given and the transversatity condition

$$\lim_{T\to\infty} \beta^T U_{C_T} K_{T+1} = 0$$

is imposed.

1 – Write the Planner problem (that will give competitive equilibrium quantities) and derive, for each $t > 0$, three equations that can be interpreted as a good market equilibrium (GM) condition, a labour market equilibrium condition (LM) and a Euler equation.

2 – Temporary equilibria are defined as the triplets (C_t, L_t, I_t) that satisfy (GM) and (LM). Why is that an interesting concept?

3 – Assume we are in period t and that an unexpected change in future U or F occurs (a "news"). Explain why such a news will directly enter only in the Euler equation, so that in period t, the economy will have to move along (GM) and (LM).

4 – Fully differentiate (GM) and (LM) (note that K_t is predetermined and therefore cannot change) and, for a given I_t, draw those two equations in the plane (C_t, L_t).

5 – Assume that the news causes an increase in I_t ($dI_t > 0$). Show that then consumption and labour will move in opposite direction. Can news create business cycle movements in that economy?

6 – Assume now that technology is given by $C = G(K, L, I)$, with $G_K > 0$, $G_{KK} < 0$, $G_L > 0$, $G_{LL} < 0$, $G_{KL} > 0$ and $G_I < 0$. Show that this formulation encompasses the previous case.

7 – Show that temporary equilibria can now display business cycle fluctuations following a news shock if G_{LI} is positive and large enough. How would you interpret that?

– ECON 0107 – MIDTERM EXAM – 2020-2021 – SOLUTION –

II BUSINESS CYCLES MOVEMENTS AT A TEMPORARY EQUILIBRIUM.

1. Planner problem: $\max \mathcal{L} = \sum \beta^t \left(u(C_t) - v(L_t) + \lambda_t \left(F(K_t, L_t) - C_t \underbrace{- K_{t+1} + (1-\delta) K_t}_{-I_t} \right) \right)$

FOC

$$u_c(C)\, F_L(K, L) = v_L(L) \quad (LM)$$

$$u_c(C) = \beta\, u_c(C')\left(1 + F_K(K', L') - \delta\right) \quad (\text{Euler})$$

$$C + I = F(K, L) \quad (GM)$$

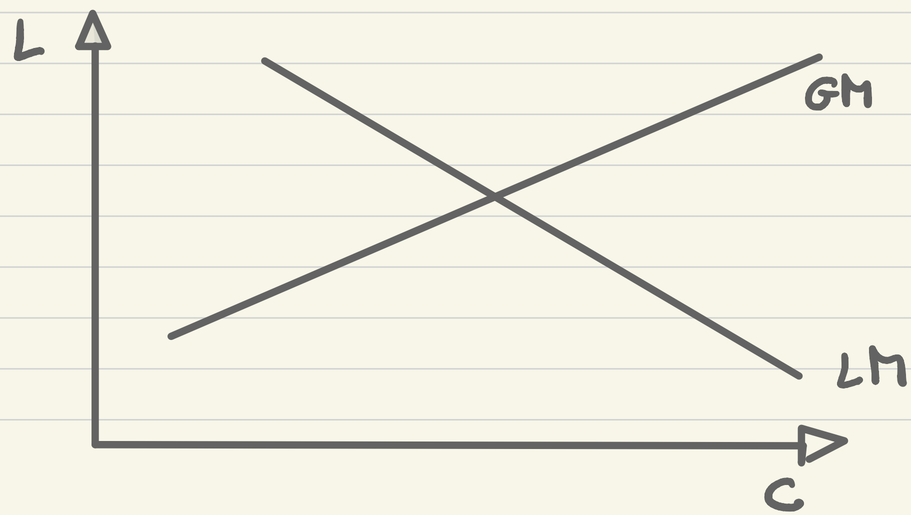
2. Temporary equilibrium: (GM), (LM) : puts restrictions on the joint movements of (C, I, L) for given expectations

3. A shock in the future does not enter directly in (LM) and (GM), but only in the right hand side of the Euler equation

(4) (GM) $\Rightarrow$ $dC + dI = F_L(K,L)\,dL$ $\Rightarrow$ $dL = \underbrace{\frac{1}{F_L}}_{(+)}(dC + dI)$

(LM) $\Rightarrow$ $u_{cc}F_L\,dC + u_c F_{LL}\,dL = v_{LL}\,dL$ $\Rightarrow$ $dL = \underbrace{\frac{u_{cc}F_L}{v_{LL} - u_c F_{LL}}}_{(-)}\,dC$

given I :

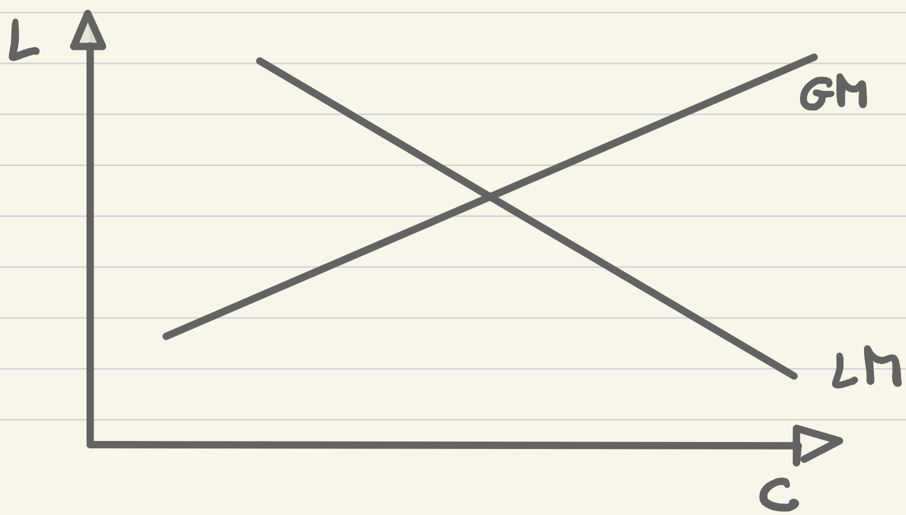


3. A shock in the future does not enter directly in (LM) and (GM), but only in the right hand side of the Euler equation

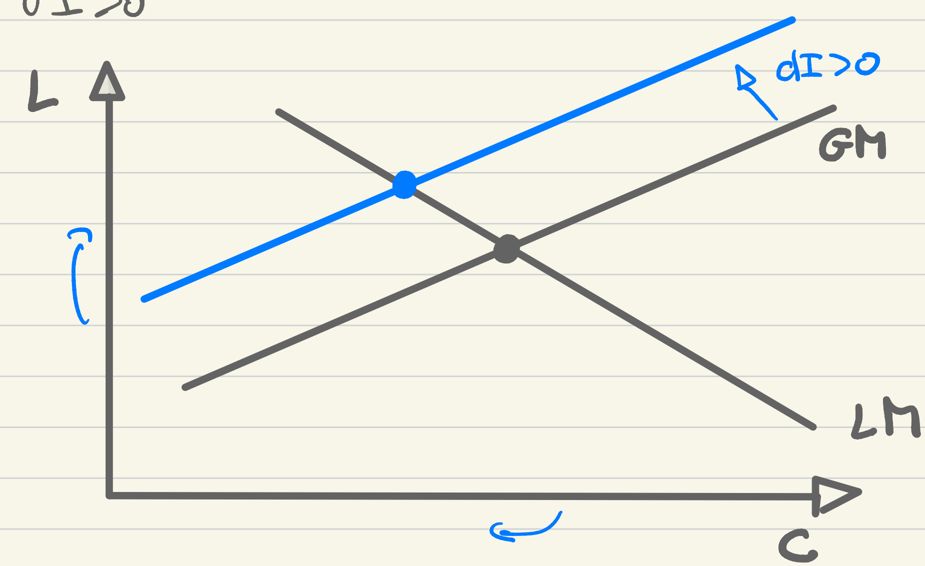
(4) (GM) $\Rightarrow$ $dC + dI = F_L(K,L)\,dL$ $\Rightarrow$ $dL = \underbrace{\frac{1}{F_L}}_{(+)}(dC + dI)$

(LM) $\Rightarrow$ $u_{cc}F_L\,dC + u_c F_{LL}\,dL = v_{LL}\,dL$ $\Rightarrow$ $dL = \underbrace{\frac{u_{cc}F_L}{v_{LL} - u_c F_{LL}}}_{(-)}\,dC$

given I :



5. assume $dI>0$



We see that, because a change in expectations moves the temporary equilibrium along the LM curve, which is decreasing if leisure and consumption are normal goods, so that C and L move in opposite direction. → news cannot create business cycles

6 - Previous case is $G_t = G(K_t, L_t, I_t) = F(K_t, L_t) - I_t$

so that $G_I = -1$, $G_{LI} = 0$

A general G function corresponds to joint production of C and I out of K and L.

7 - Temporary equilibrium is now

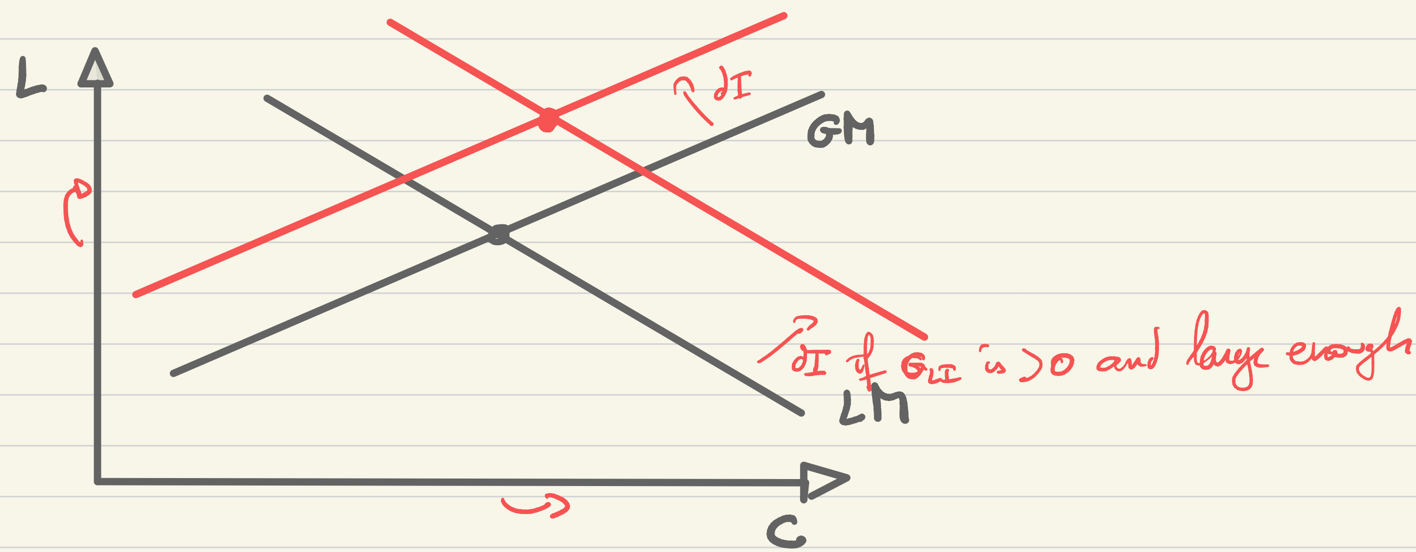
(GM) $C = G(K, L, I) \Rightarrow dC = G_L\, dL + G_I\, dI$

(LM) $u_c(C)\, G_L(K, L, I) = V_L(L) \Rightarrow u_{cc} G_L\, dC + u_c G_{LL}\, dL + u_c G_{LI}\, dI = V_{LL}\, dL$

↳ (GM): $dL = \frac{1}{G_L}(dC + dI)$

(LM) $dL = \dfrac{u_{cc} G_L}{V_{LL} - u_c G_{LL}}\, dC + \dfrac{u_c G_{LI}}{V_{LL} - u_c G_{LL}}\, dI$

new term here



We see that one can have now a joint increase in C, L and I (or a joint decrease if the news causes I to fall)

What does $G_{LI} \gg 0$ mean? Marginal productivity of labor in the consumption good sector increases when the economy produces more investment.

This is a case of 'economies of scope' in a joint production setup

[a bit similar to 'increasing returns', but here G stays concave]

Consider a model with a continuum of households who are subject to idiosyncratic returns on their investment. In particular, each household can invest in a risk-free bond with rate of return r and a risky capital whose rate of return is random and variable across individuals. Assume that households can borrow using the risk-free bond while their investment in risky capital has to be positive. More specifically, the household problem reads:

$$\max_{c_t,\phi_t} \quad \sum_{t=0}^{\infty} \beta^t \ln c$$
$$s.t. \quad A_{t+1} = (A_t - c_t)((1+r)\phi_t + (1+z_t)(1-\phi_t)) + e,$$

where A is cash-on-hand and e is a constant endowment. The household decides on consumption each period, c_t, and the fraction of savings, ϕ_t, invested in the risk-free asset. The fraction $(1-\phi_t)$ of savings goes into capital with stochastic return z_t (iid across time and households).

1 – What is the natural borrowing limit for the risk-free asset?

e/r. Think of bond holdings being shifted by this limit such that you only have to consider non-negative asset holdings.

2 – Write down the Bellman equation corresponding to the household problem. What is the state variable?

$$V(A_t) = \max_{c_t,\phi} [\ln c_t + \beta E_t V((A_t - c_t)((1+r)\phi + (1+z_t)(1-\phi)))]$$

The problem is autonomous so we write the current value Bellman equation with time independent value function V.

3 – Derive the first-order conditions. What is the interpretation?

$$c_t; \quad 1/c_t - \beta E_t V'(A_{t+1})((1+r)\phi + (1+z_t)(1-\phi)) = 0$$
$$\phi_t; \quad E_t V'(A_{t+1})(A_t - c_t)(r - z_t) = 0$$

4 – Guess and verify the solution to the Bellman equation. (Hint: ln utility!) Describe the properties of the optimal consumption and portfolio policy.

Guess and verify: $V(A_t) = a \ln A_t + B$ for some constants a and B. Substitute into the FOCs to get:

$$c_t = \frac{A_t}{1+\alpha\beta}$$
$$E_t \frac{(r - z_t)}{((1+r)\phi_t + (1+z_t)(1-\phi_t))} = 0$$

The latter implies that ϕ_t is constant since z_t is iid. Now solve for constant a by substituting the solutions to the FOCs and the guess into the Bellman equation. This yields after same algebra:

$$a \ln A_t + B = (1+\alpha\beta) \ln A_t + k,$$

where k is some constant. This also verifies that the guess worked.

5 – Discuss the implications for aggregate asset holdings. Under what conditions does it converge?

Asset holding evolve according to

$$A_{t+1} = \beta A_t((1+r)\phi + (1+z_t)(1-\phi)) + e,$$
$$A_{t+1} = \beta A_t R + e.$$

Hence, aggregate assets follow a random walk with positive drift for $\beta R = 1$. $\beta R < 1$ is necessary for convergence.

6 – Now suppose that we have a closed economy consisted of individuals like above who are symmetric and can trade the risk-free bond. Define an equilibrium in this economy.

A recursive competitive equilibrium is a value function V; policy functions c, ϕ for the household; an interest rate r; and, a stationary measure such that:

1. given interest rate r, the policy functions c, ϕ solve the household's problem given by the Bellman equation and V is the associated value function
2. the asset market clears: $0 = \int_A \phi A - e/r$, where bond holdings are shifted by e/r to be non-negative reals.

7 – Now suppose you would like to analyse aggregate shocks, e.g. a shock to the variance of capital returns. What state variables are needed to describe the evolution of the economy? How would you compute the solution?

Key is that the policy functions are linear. Hence, the distribution of households does not matter for predicting the interest rate. The state variables only consist of aggregate states. This includes $\bar{A}$ and potentially other aggregate states like inertia in the variance of returns. Different from Krusell&Smith, one can therefore compute the RE equilibrium using standard methods.

IV – McCall Model

Consider a worker who draws every period a job offer from a cumulative distribution $F(w)$ to work forever at wage w. Each time the worker draws a job offer a cost C incurs, where $0 < C < E[w]$. The worker's objective is to maximize the expected value of $w - nC$, where w is the accepted wage offer and n is the number of job offers the worker has drawn. Let V denote the expected value of $w - n'C$ of a worker who has just rejected a job, where n' is the number of jobs the worker will draw from that point on.

1 – Explain why the worker accepts a job offering $\tilde{w}$ if $\tilde{w} > V$, and rejects its if $\tilde{w}$ if $\tilde{w} < V$.

Note that $V = E[w - Cn'] = E[w] - CE[n']$ and is the expected wage the worker will eventually accept if she searches more minus the expected cost of further searching. The exp. cost of further searching is the expected number of jobs to be sampled multiplied by the (known) cost of sampling each job. Thus V can be interpreted as the expected value of further searching. If the worker is offered a job that pays a wage of $\bar{w}$, where $\bar{w}$ exceeds the expected value of further searching, it is optimal to stop searching and take the job. If the wage offer is less, it is optimal to reject and continue searching.

2 – Explain why V satisfies $V = F(V)V + \int_{w=V}^{\infty} wf(w)dw - C$, where f is density function associated with F.

Not that it can be rewritten as:

$$V = \frac{\int_{w=V}^{\infty} wf(w)dw}{1 - F(V)} - \frac{C}{1 - F(V)}$$

First term represents the expected wage conditional on that wage being greater than the reservation wage of V. Second term represents the expected cost of sampling jobs. Thus $V = E[w] - CE[n']$ must satisfy this equation.

3 – Does a searcher ever want to accept a job that has been previously rejected?

No. V is constant and so is $\bar{w}$.

★

Now, suppose w is distributed uniformly on $[\mu - a, \mu + a]$ and that $C < \mu$. This implies that $f(w) = 1/2a$ and $F(w) = \frac{w-(\mu-a)}{2a}$.

4 – Find V in terms of μ, a, and C.

Start from

$$V = \left[\frac{V - (\mu - a)}{2a}\right] V + \int_{w=V}^{\mu+a} (w/2a)dw - C$$

The integral is: $1/4a[(\mu + a)^2 - V^2]$

Plug back to get:

$$V^2 - 2(\mu + a)V + (\mu + a)^2 - 4aC = 0$$

Using the quadratic formula gives:

$$V = \frac{2(\mu + a) + / - 4\sqrt{aC}}{2}$$

Ignore $V > \mu + a$ since $\mu + a$ is the highest wage, which gives the solution for V.

5 – How does an increase in a affect V? Explain intuitively.

$$\delta V/\delta a = 1 - \sqrt{(C/a)}$$

With $C < a$, a rise in a increases the reservation wage. Option value effect.