2023-2024 – Econ 0107 – Macroeconomics I

Lecture 1 : Complete Markets

(Chapter 8 in Ljunqvist & Sargent 4th edition)

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1. Time 0 versus Sequential Trading

- This chapter describes competitive equilibria for a pure exchange infinite horizon economy with stochastic endowments.
- This economy is useful for studying risk sharing, asset pricing, and consumption.
- Two market structures:
  - ARROW-DEBREU structure with complete markets in dated contingent claims all traded at time 0,
  - sequential-trading structure with complete one-period ARROW securities
- These two imply different assets and timings of trades, but have identical consumption allocation.

2. Endowments and Preferences

We start by characterizing:

1. The physical (and stochastic) properties of the resources available to the agents.
2. Agents preferences.

2. Endowments and Preferences

Histories

- We consider a stochastic event or state variable $s_t \in S$.
- Let's define the history of events up to time t as:

$$s^t = \{s_0, s_1, ..., s_t\}$$

- The unconditional probability of observing state s_t in history s^t is

$$\pi_t(s^t)$$

- Conditional probabilities, for history s^t conditional on history s^τ, $\tau < t$, are given by:

$$\pi_t(s^t|s^\tau)$$

- Note: we will not immediately assume a Markov structure.
  - × Markov structure = memoryless property of a stochastic process, which means that its future evolution is independent of its history.
  - × While useful for some results, it is not necessary for others.
- Trade after observing s_0, therefore $\pi_0(s_0) = 1$.

Endowments

- We assume the economy is populated by I agents, denoted by index $i = 1, 2, ..., I$.
- Each agent receives an endowment which is a function of the random history s^t:

$$y_t^i(s^t).$$

- Each household will be associated with a consumption plan

$$c_i = \{c_t^i(s^t)\}_{t=0}^{\infty}.$$

- Consumption plans will be ordered using the following function:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t) = E_0 \sum_{t=0}^{\infty} \beta^t u[c_t^i(s^t)].$$

- Regularity conditions on $u[.]$.
  - × Increasing, concave and smooth.
  - × Inada condition.

$$\lim_{c \downarrow 0} u'(c) = +\infty$$

- Consumers share probabilities (views of the world)

- All histories are fully observable, verifiable and contractable upon.

Definition 1

*An **allocation** of resources is a collection of consumption plans* $C = \{c_i\}_{i=1}^{I}$.

Definition 2

*An allocation of resources is **feasible** if:*

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \quad \forall t \quad and \quad \forall s^t.$$

- Notice: we are not considering storage.

3. Alternative Trading Arrangements

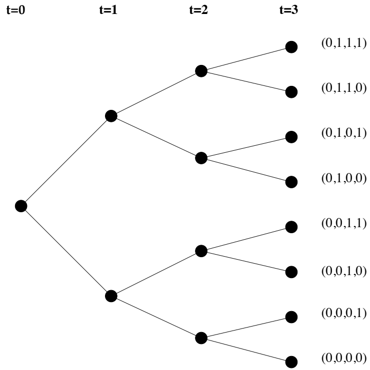
- We will consider two different trading arrangements.
- We will then show their equivalence.

1. We will first consider a situation where all trades happen at time 0.
   - × Agents will trade state and history contingent claims.
   - × Once the trade is done, markets close for subsequent periods.
   - × Contracts are enforced in subsequent periods and commitments honored.
2. We will then consider a sequential trading framework.
   - × In each period agents will trade one-period contingent claims.

- Those two trading arrangement
  - × share the same consumption allocations
  - × share the property that allocations depend only on aggregate resources at each date and on a a time-invariant wealth distribution.

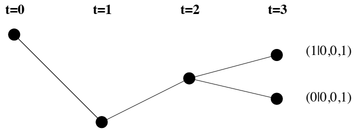
3. Alternative Trading Arrangements

Figure 1: Time 0 ARROW-DEBREU trades: all possible histories are considered at time 0.



The ARROW-DEBREU commodity space for a two-state Markov chain. At time 0, there are trades in time t = 3 goods for each of the eight nodes that signify histories that can possibly be reached starting from the node at time 0.

Figure 2: Sequential trades in 1-period ARROW securities



The commodity space with ARROW securities. At date t = 2, there are trades in time 3 goods for only those time t = 3 nodes that can be reached from the realized time t=2 history (0,0,1).

4. Pareto Problem

Definition 3

An allocation is said to be Pareto*-optimal if any re-allocation that makes one household strictly better off also makes one or more other households worse off.*

Definition 4

An allocation is said to be efficient if it is Pareto*-optimal.*

- We can construct efficient allocations from a Pareto problem:
- A social planner maximizes the weighted average of individual utilities using some arbitrary non-negative Pareto weights $\lambda_i, i = 1, ..., I$.

$$W = \sum_{i=1}^{I} \lambda_i U(c^i) \quad \textit{subject to}$$

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \;\; \forall t \;\; \textit{and} \;\; \forall s^t.$$

4. Pareto Problem

- Consider the Lagrangian for this problem:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u(c_t^i(s^t)) \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] \right\}$$

  where $\theta_t(s^t)$ are (non-negative) Lagrange multipliers associated to the feasibility constraints relevant in each state and time.

- First order condition w.r.t. $c_t^i(s^t)$:

$$\beta^t u'(c_t^i(s^t)) \pi_t(s^t) = \lambda_i^{-1} \theta_t(s^t) \quad \forall i, t, s^t.$$

- Considering the ratio of this condition to that for consumer 1:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i}$$

4. Pareto Problem

- Solving for $c_t^i(s^t)$:

$$c_t^i(s^t) = u'^{-1}\left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))\right). \qquad (\star)$$

- Substituting in the feasibility constraint:

$$\underbrace{\sum_i u'^{-1}\left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))\right) = \sum_i y_t^i(s^t).}_{\text{one equation, one unknown}} \qquad (\star\star)$$

Proposition 1

An efficient allocation is a function of the realized aggregate endowment and depends neither on the specific history leading up to that outcome nor on the realizations of individual endowments.

$$c_t^i(s^t) = c_\tau^i(\bar{s}^\tau) \;\; \forall\, s^t \;\; \textit{and} \;\; \bar{s}^\tau \;\; \textit{such that} \;\; \sum_j y_t^j(s^t) = \sum_j y_\tau^j(\bar{s}^\tau).$$

4. PARETO Problem

- Solving for $c_t^i(s^t)$:

$$c_t^i(s^t) = u'^{-1}(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))). \qquad (\star)$$

- Substituting in the feasibility constraint:

$$\underbrace{\sum_i u'^{-1}(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))) = \sum_i y_t^i(s^t).}_{\text{one equation, one unknown}} \qquad (\star\star)$$

- To solve for P.O. allocations, we will for each realization of history s^t:
  - × Solve $(\star\star)$ for $c_t^1(s^t)$
  - × Solve $(\star)$ for all $i \neq 1$ $c_t^i(s^t)$

Notice:

- Pareto weights can be normalized (only ratios matter) $\rightsquigarrow$ we can impose $\sum_i \lambda_i = 1$.
- Pareto weights are time invariant.
- Relative marginal utilities depend on Pareto weights only, so that they are time invariant.
- Consumer's i relative share of the aggregate endowment varies with his Pareto weight λ_i.
- So far we have described the allocations, not the specific trades made to achieve those allocations.

5. Time-0 trading: ARROW-DEBREU securities.

- Here we describe a particular market organisation that reaches as PARETO-efficient allocation.
- At time zero consumers trade entitlements to state-contingent consumption streams.
- The price of a security that delivers one unit of consumption at time t if the history up to that point has been s^t is $q_t^0(s^t)$.
- In other words, $q_t^0(s^t)$ is the price at time 0 of time t consumption contingent on history s^t, in terms of an abstract unit of account.
- The budget constraint for a consumer entering the time 0 market will then be:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t).$$

- Consumer i will maximize utility given this budget constraint.
- Notice: there is a single budget constraint because all trades occur at time 0.

5. Time-0 trading: Arrow-Debreu securities.

- For a generic household i

$$\max U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) \qquad (\mu_i)$$

- First order condition will be:

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \mu_i q_t^0(s^t).$$

μ_i is the Lagrange multiplier associated with budget constraint.

- Given intertemporal separability of preferences,

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \beta^t u'[c_t^i(s^t)] \pi_t(s^t) = \mu_i q_t^0(s^t).$$

5. Time-0 trading: Arrow-Debreu securities.

Definition 5

A **price system** *is a sequence of functions* $\{q_t^0(s^t)\}_{t=0}^{\infty}$.

Definition 6

An **allocation** *is a list of sequence of functions* $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$.

Definition 7

A **competitive equilibrium** *is a feasible allocation and a price system such that, given the price system, the allocation solves each household's problem.*

5. Time-0 trading: Arrow-Debreu securities.

- Notice that:

$$\beta^t u'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$

implies:

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^j(s^t)]} = \frac{\mu_i}{\mu_j}, \quad \forall i, j$$

- An equilibrium, therefore, solves:
  1. The aggregate feasibility constraint.
  2. The individual budget constraint for all individuals.
  3. The condition on the ratios of marginal utilities.

5. Time-0 trading: Arrow-Debreu securities.

- Notice that one can solve the equation for the ratio of marginal utilities for $c_t^i(s^t)$ as a function of:
  1. consumption of individual 1;
  2. the ratio of Lagrange multipliers of individuals i and 1

$$c_t^i(s^t) = u'^{-1}\left\{ u'[c_t^1(s^t)]\frac{\mu_i}{\mu_1} \right\}$$

- Substituting in the aggregate feasibility constraints gets :

$$\sum_i u'^{-1}\left\{ u'[c_t^1(s^t)]\frac{\mu_i}{\mu_1} \right\} = \sum_i y_t^i(s^t).$$

- This equation can be solved for $c_t^1(s^t)$.
- Notice that the right-hand-side is the aggregate endowment.

Proposition 2

The competitive equilibrium allocation is a function of the realized aggregate endowment and does not depend on time t or on the specific history or on the cross section distribution of endowments: $c_t^i(s^t) = c_\tau^i(\widetilde{s}^\tau)$ *for all histories* s^t *and* $\widetilde{s}^\tau$ *such that* $\sum_j y^j(s^t) = \sum_j y^j(\widetilde{s}^\tau)$.

- Remark: $\left\{\frac{\mu_j}{\mu_1}\right\}_{j=2}^{I}$ and aggregate endowment in t are what determine the distribution of consumption at date t.

- Having found the equilibrium, we can use expressions such has:

$$\beta^t u'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$

  to determine the equilibrium price $q_t^0(s^t)$.
- Notice that prices are stochastic processes.
- Price units are arbitrary so that we can for example normalize one price to unity.
- A possibility is to set $q_0^0(s_0) = 1$.
- This implies $\mu_i = u'[c_0^i(s_0)]$.

- An important property of the competitive equilibrium is that it is Pareto optimal.
- This can be seen if set the Pareto weights in the Social Planner problem so that:

$$\lambda_i = 1/\mu_i, \quad \forall i$$

- $\theta_t(s^t) = q_t^0(s^t)$.
- The coincidence of competitive equilibria and Pareto optimality is behind the First and Second fundamental theorems of welfare economics.

Theorem 1

First Welfare Theorem: *Any Competitive Equilibrium is* Pareto *optimal.*

Theorem 2

Second Welfare Theorem: *Under some regularity conditions, any* Pareto *optimal allocation can be sustained as a competitive equilibrium.*

5. Time-0 trading: Arrow-Debreu securities.

Equilibrium computation.

To compute the equilibrium on can use the Negishi *algorithm* to determine μ_i/μ_1.

1. Fix an arbitrary (positive) value for μ_1. Guess some initial positive value for the other μ_i. Compute:

$$c_t^i(s^t) = u'^{-1}\left\{u'[c_t^1(s^t)]\frac{\mu_i}{\mu_1}\right\}.$$

$$\sum_i u'^{-1}\left\{u'[c_t^1(s^t)]\frac{\mu_i}{\mu_1}\right\} = \sum_i y_t^i(s^t).$$

2. Solve for equilibrium prices $q_t^0(s^t)$:

$$q_t^0(s^t) = \beta^t u'[c_t^i(s^t)]\pi_t(s^t)\mu_i^{-1}$$

3. Check the budget constraint for every $i = 1, ..., I$.
   - × For those i where c_i is too high, raise μ_i.
   - × For those i where c_i is too low, lower μ_i.
4. Iterate previous steps until convergence.

- In general, the computation of equilibrium is a difficult problem.
- Typically, the equilibrium prices depend on the wealth distribution (*i.e.* all the individual endowment streams)
- There are some cases (particular preferences or endowment processes) where the computation is simpler, as we do not need to iterate on PARETO weights.

- The model we have described has important implications for risk sharing.
- Individuals with concave utility will want to smooth consumption over time.
- In the model we have studied, even without storage possibilities, smoothing is possible if individual shocks are not perfectly correlated.
- Indeed we will consider examples in which individuals can smooth quite a bit.

- Suppose utility is CRRA:

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0$$

- Then in the complete markets equilibrium:

$$\frac{[c_t^i(s^t)]^{-\gamma}}{\mu_i} = \frac{[c_t^j(s^t)]^{-\gamma}}{\mu_j}, \quad \forall i, j$$

$$\implies c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}, \quad \forall i, j$$

- Individual consumptions are perfectly correlated.

- The model implies that the consumption of different agents varies in the same proportion so that the ratio stays constant over time.
- There is extensive cross-period and cross-states risk sharing.
- The factor of proportionality is given by the ratio of multipliers μ_i/μ_j, or, alternatively, by the ratio of PARETO weights in the social planner program.
- From the FOC of the social planner program, we can notice that the only thing that determines individual consumption is aggregate shocks:

$$u'[c_t^i(s^t)]\lambda_i\beta^t\pi_t(s^t) = [c_t^i(s^t)]^{-\gamma}\lambda_i\beta^t\pi_t(s^t) = \theta_t(s^t).$$

- In this example, we can easily compute equilibrium prices. Using

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}, \quad \forall i, j$$

- and the price formula derived earlier

$$\beta^t u'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$

- we obtain

$$q_t^0(s^t) = \mu_i^{-1}\alpha_i^{-\gamma}\beta^t(\overline{y}_t(s^t))^{-\gamma}\pi_t(s^t)$$

where $c_t^i(s^t) = \alpha_i \overline{y}_t(s^t)$, $\overline{y}_t(s^t) = \sum_j y_t^i(s^t)$, $\alpha_i = \left(\sum_j \left(\frac{\mu_i}{\mu_j}\right)^{\frac{1}{\gamma}}\right)^{-1}$

- We have one normalisation choice, which amount to setting $\mu_i^{-1}\alpha_i^{-\gamma}$ for one i to an arbitrary positive number.
- For example $\mu_1^{-1}\alpha_1^{-\gamma} = 1$

- We then compute equilibrium as follows:
- Prices are obtained from

$$q_t^0(s^t) = \beta^t(\overline{y}_t(s^t))^{-\gamma}\pi_t(s^t)$$

- Using those prices and consumer's i budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t),$$

- we obtain the consumption share α_i as its share of total wealth evaluated at equilibrium prices:

$$\alpha_i = \frac{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t)}{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\overline{y}_t(s^t)}$$

Empirical implications

- Using the Social planner FOC
- We have

$$u'[c_t^i(s^t)]\lambda_i\beta^t\pi_t(s^t) = [c_t^i(s^t)]^{-\gamma}\lambda_i\beta^t\pi_t(s^t) = \theta_t(s^t)\Big[= q_t^0(s^t)\Big].$$

- Taking logs:

$$-\gamma\log(c_t^i(s^t)) + \log(\lambda_i) + \log(\beta^t\pi_t(s^t)) = \log(\theta_t(s^t)).$$

$$-\gamma\log(c_t^i(s^t)) + \log(\lambda_i) = \log(\theta_t(s^t)) - t\log(\beta) - \log(\pi_t(s^t)).$$

- Taking this equation at two different points in time (t and τ) and subtracting one from the other:

$$\begin{aligned} -\gamma[\log(c_t^i(s^t)) - \log(c_\tau^i(s^\tau)] &= \log(\theta_t(s^t)) + \log(\pi_t(s^t)) - \log(\theta_\tau(s^\tau)) \\ &- \log(\pi_\tau(s^\tau)) - (t-\tau)\log(\beta) = \nu_{t,\tau}. \end{aligned}$$

- No index i in $\nu_{t,\tau}$.

Empirical implications

- This equation gives an empirical prediction:

$$[\log(c_t^i(s^t)) - \log(c_\tau^i(s^\tau)] = -\frac{1}{\gamma}\nu_{t,\tau}$$

- Note:
  - × Strong empirical test: individual consumption growth only depends on aggregate growth.
  - × Townsend (1994) and others used this strategy explicitly.
  - × There are no residuals in this equation
    - Measurement error can be added.
    - Taste shocks.

- There are other examples in Chapter 8 of Ljunqvist & Sargent's book that you have to work by yourself.

7. Primer on Asset pricing

- Many asset-pricing models assume complete markets and price an asset by breaking it into a sequence of history-contingent claims, evaluating each component of that sequence with the relevant "state price deflator" $q_t^0(s^t)$, and then adding up those values.
- The asset is viewed as redundant, in the sense that it offers a bundle of history-contingent dated claims, each component of which has already been priced by the market.

- Let $\{d(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t, state s^t consumption, where $d(s_t)$ is a measurable function of s_t.
- The price of an asset paying the owner this stream must be

$$p_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d(s^t)$$

- This can be understood as an arbitrage equation.

- A riskless consol offers for sure one unit of consumption at each period, i.e. $d_t(s^t) = 1$ for all t and s^t.
- The price is

$$p_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)$$

- Consider a sequence of strips of returns on the riskless consol.
- The time-t strip is the return process $d_\tau = 1$ if $\tau = t \geq 0$, and 0 otherwise.
- The price of time-t strip at 0 is

$$p_t^0 = \sum_{s^t} q_t^0(s^t)$$

- Consider the stream of dividends $\{d(s_t)\}_{t\geq 0}$
- For $\tau \geq 1$, suppose that we strip off the first $\tau - 1$ periods of the dividend and want to get the time-0 value of the dividend stream $\{d(s_t)\}_{t\geq\tau}$.
- Let $p_\tau^0(s^\tau)$ be the time-0 price of an asset that entitles the dividend stream $\{d(s_t)\}_{t\geq\tau}$ if history s^τ is realized:

$$p_\tau^0(s^\tau) = \sum_{t\geq\tau} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} q_t^0(\widetilde{s}^t) d(\widetilde{s}^t)$$

- Let us convert this price into units of time τ, state s^τ by dividing by $q_\tau^0(s^\tau)$:

$$p_\tau^\tau(s^\tau) = \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t\geq\tau} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} \frac{q_t^0(\widetilde{s}^t)}{q_\tau^0(s^\tau)} d(\widetilde{s}^t)$$

- Notice that for all consumers i

$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u'[c_t^i(s^t)]\pi(s^t)}{\beta^\tau u'[c_\tau^i(s^\tau)]\pi(s^\tau)} = \beta^{t-\tau} \frac{u'[c_t^i(s^t)]}{u'[c_\tau^i(s^\tau)]} \pi(s^t | s^\tau) \qquad (\star)$$

- $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t, state s^t in terms of the date-τ, state-s^τ consumption good.
- The price at τ in history s^τ for the tail asset is

$$p_\tau^\tau(s^\tau) = \sum_{t \geq \tau} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} q_t^\tau(\widetilde{s}^t) d(\widetilde{s}^t)$$

- This tail asset formula is useful if one wants to create in a model a time series of equity prices: an equity purchased at time τ entitles the owner to the dividends from time τ forward, and the price is given by the above formula.

$$q_t^\tau(s^t) = \beta^{t-\tau}\frac{u'[c_t^i(s^t)]}{u'[c_\tau^i(s^\tau)]}\pi(s^t|s^\tau) \qquad (\star)$$

- We have shown that $c_t^i(s^t)$ are not history dependant.
- Then the relative price in $(\star)$ is not history dependent.

Proposition 3

The equilibrium price of date-$t \geq 0$, state-s^t consumption good expressed in terms of date τ ($0 \leq \tau \leq t$), state s^τ consumption good is not history dependent:

$$q_t^\tau(s^t) = q_t^j(\widetilde{s}^k)$$

for $j, k \geq 0$ such that $t - \tau = k - j$ and $[s_t, s_{t-1}, \ldots, s_\tau] = [\widetilde{s}_k, \widetilde{s}_{k-1}, \ldots, \widetilde{s}_j]$.

- The one-period version of equation ($\star$) is

$$q^{\tau}_{\tau+1}(s^{\tau+1}) = \beta \frac{u'(c^i_{\tau+1})}{u'(c^i_{\tau})} \pi(s_{\tau+1}|s_{\tau})$$

- The RHS is the one-period *pricing kernel* at time τ.
- The price at time τ in state s^{τ} of a claim to a random payoff $\omega(s_{\tau+1})$ is given, using the pricing kernel, by

$$\begin{aligned} p^{\tau}_{\tau}(s^{\tau}) &= \textstyle\sum_{s_{\tau+1}} q^{\tau}_{\tau+1}(s^{\tau+1})\omega(s_{\tau+1}) \\ &= E_{\tau}\left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})}\omega(s_{\tau+1})\right] \end{aligned}$$

where superscripts i and dependence to s_{τ} have been deleted.
- Let denote the one-period gross return on the asset by $R_{\tau+1} = \omega(s_{\tau+1})/p^{\tau}_{\tau}(s^{\tau})$. Then, for any asset, the above equation implies

$$1 = E_{\tau}\left[\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} R_{\tau+1}\right]$$

$$1 = E_\tau \left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} R_{\tau+1} \right]$$

- The term $m_{\tau+1} = \beta \frac{u'(c_{\tau+1})}{u'(c_\tau)}$ is a *stochastic discount factor*.
- That equation can be understood as a restriction on the conditional moments of returns and $m_{\tau+1}$.
- Applying the law of iterated expectations to the above equation, one gets the unconditional moments restriction:

$$1 = E \left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} R_{\tau+1} \right]$$

8. Sequential trading.

- After considering the time-0 trading we consider sequential trading.
- For this, the one-period formula we have derived earlier will be crucial.
- We will show that the allocations that prevail under complete markets, as described by the zero-period trade in Arrow-Debreu state contingent assets, can be replicated with one-period securities.
- We will consider the household value function and write it as a function of a crucial state variable.

8. Sequential trading.

- We define as 'household wealth' the value (at a point in time) of all the claims 'owned' by a household net of its liabilities.

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)].$$

- Budget constraint at equality implies $\Upsilon_0^i(s^0) = 0$
- Notice that at time t, given history s^t, we can ignore all the claims and liabilities that do not correspond to that particularly history.
- At time t of history s^t, typically $\Upsilon_t^i(s^t) \neq 0$, but...
- ... feasibility implies

$$\sum_{i=1}^{I} \Upsilon_t^i(s^t) = 0, \quad \forall t, s^t.$$

- When considering the time-0 equilibrium we impose the intertemporal budget constraint.
- In the case of sequential equilibria with infinitely lived consumers we need to avoid 'Ponzi schemes'.
- This is equivalent to assuming the consumer will have to repay her debts in any state of the world.
- We will impose a *natural debt limit*, given, for every history, by the sum of future endowments.

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau).$$

- Each consumer will not be able to promise to pay, for each history s^t more than $A_t^i(s^t)$, which corresponds to zero consumption.

- Suppose our consumers are operating in a sequence of markets in one-period-ahead state-contingent claims.
  - At time t consumers trade on claims contingent on s_{t+1}.
- Let $\widetilde{a}_t^i(s^t)$ be claims brought into time t.
- $\widetilde{Q}_t(s_{t+1}|s^t)$ is the price (at time t) of one unit of time $t+1$–consumption, contingent on s_{t+1}, when history has been s^t.
- Budget constraint:

$$\widetilde{c}_t^i(s^t) + \sum_{s_{t+1}} \widetilde{a}_{t+1}^i(s_{t+1}, s^t)\widetilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \widetilde{a}_t^i(s^t).$$

- Debt limits:

$$-\widetilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}).$$

  This is a borrowing constraint faced by each individual in each state of the world.

- The Lagrangian for the consumer problem will be given by:

$$L^i = \sum_{t=0}^{\infty} \sum_{s^t} \Bigg\{ \beta^t u(\widetilde{c}_t^i(s^t))\pi_t(s^t)$$

$$+ \eta_t^i(s^t)\left[y_t^i(s^t) + \widetilde{a}_t^i(s^t) - \widetilde{c}_t^i(s^t) - \sum_{s_{t+1}} \widetilde{a}_{t+1}^i(s_{t+1}, s^t)\widetilde{Q}_t(s_{t+1}|s^t) \right]$$

$$+ \sum_{s_{t+1}} \widetilde{\nu}_t^i(s_{t+1}, s^t)\left[A_{t+1}^i(s^{t+1}) + \widetilde{a}_{t+1}^i(s^{t+1}) \right] \Bigg\}$$

for a given initial wealth $\widetilde{a}_0^i(s_0)$.

- First Order Conditions w.r.t. $\widetilde{c}_t^i(s^t)$ and $\widetilde{a}_{t+1}^i(s_{t+1}, s^t)$:

$$\beta^t u'(\widetilde{c}_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0.$$

$$-\eta_t^i(s^t)\widetilde{Q}_t(s_{t+1}|s^t) + \widetilde{\nu}_t^i(s_{t+1}, s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

- Because of the Inada conditions, the debt limit constraint will not be binding at the optimum.

$$\implies \quad \widetilde{\nu}_t^i(s_{t+1}, s^t) = 0, \quad \forall t, s^t, s_{t+1}.$$

$$-\eta_t^i(s^t)\widetilde{Q}_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

- $$\beta^t u'(\widetilde{c}_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0.$$
$$-\eta_t^i(s^t)\widetilde{Q}_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$
- Substituting the first equation in the second yields:
$$\widetilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\widetilde{c}_{t+1}^i(s^{t+1}))}{u'(\widetilde{c}_t^i(s^t))}\pi_t(s_{t+1}|s^t), \quad \forall t, s^t, s_{t+1}.$$

8. Sequential trading.

Definition 8

A **distribution of wealth** *is a vector* $\overrightarrow{\widetilde{a}}_t(s^t) = \{\widetilde{a}^i_t(s^t)\}_{i=1}^I$ *such that* $\sum_i \widetilde{a}^i_t(s^t) = 0$.

Definition 9

A **sequential-trading competitive equilibrium** *is an initial distribution of wealth* $\overrightarrow{\widetilde{a}_0}(s_0)$, *an allocation* $\{\widetilde{c}^i\}_{i=1}^I$, *and pricing kernels* $\widetilde{Q}_t(s_{t+1}|s^t)$ *such that:*

1. *for all* i, *given* $\widetilde{a}_0(s_0)$ *and the pricing kernels, the consumption allocation* $\widetilde{c}^i$ *solves household* i *consumption problem.*
2. *for all realizations of* $\{s^t\}_{t=0}^\infty$ *the households' consumption allocation and implied asset portfolios satisfy:*

$$\sum_i \widetilde{c}^i_t(s^t) = \sum_i y^i_t(s^t),$$

$$\sum_i \widetilde{a}^i_{t+1}(s_{t+1}, s^t) = 0.$$

9. Equivalence of allocations under time-zero trading and sequential trading.

- With an appropriate choice of pricing kernel, one can show that a competitive equilibrium allocation of the complete markets model with time 0 trading is also a sequential-trading competitive equilibrium allocation ...
- ..., provided that we properly choose the initial distribution of wealth $\widetilde{a}_0(s_0)$.
- Remark: we must choose an initial distribution of wealth because it is *not* an endogenous variable.

9. Equivalence of allocations under time-zero trading and sequential trading.

Pricing kernel choice

- Consider the price $q_t^0(s^t)$ in the ARROW-DEBREU equilibrium.
- Consider the sequence of pricing kernels given by:

$$\widetilde{Q}_t(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_{t+1}^t(s^{t+1}).$$

- Consider the ARROW-DEBREU equilibrium consumption allocation $\{c_t^i(s^t)\}$.
- Given the pricing kernels just defined, it follows that:

$$\frac{\beta u'[c_{t+1}^i(s^{t+1})]\pi(s^{t+1}|s^t)}{u'[c_t^i(s^t)]} = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \widetilde{Q}_t(s_{t+1}|s^t)$$

- It follows that the ARROW-DEBREU time-0 trading allocation satisfies the Euler equation (first order condition) for the sequential trading problem when the pricing kernel are those defined.

- Sequential trading allocations are indexed by the initial wealth distribution.
- We therefore need to choose a wealth distribution that generates the Arrow-Debreu allocation.
- We conjecture that the initial wealth distribution is the null vector.
- Using the intertemporal budget constraints we prove that the portfolio choices induced imply the same sequence of consumption and that this is optimal.

9. Equivalence of allocations under time-zero trading and sequential trading.

Initial wealth distribution

- Suppose that at time t and history s^t , household i chooses the following asset portfolio :

$$\widetilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)].$$

  where the consumption sequence is the ARROW-DEBREU equilibrium one.

- The value of that portfolio is

$$\begin{aligned} \sum_{s_{t+1}} \widetilde{a}^i_{t+1}(s_{t+1}, s^t)\widetilde{Q}_t(s_{t+1}|s^t) &= \sum_{s^{t+1}|s^t} \Upsilon^i_{t+1}(s^{t+1})q^t_{t+1}(s^{t+1}) \\ &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} [c^i_\tau(s^\tau) - y^i_\tau(s^\tau)]q^t_\tau(s^\tau). \qquad (\dagger) \end{aligned}$$

- Notice $q^{t+1}_\tau(s^\tau)q^t_{t+1}(s^{t+1}) = \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = q^t_\tau(s^\tau), \; (\tau > t).$

9. Equivalence of allocations under time-zero trading and sequential trading.

- Let's show that the portfolio sequence $\left\{\widetilde{a}^i_{t+1}(s_{t+1}, s^t)\right\}$ is affordable
- Take BC:

$$\widetilde{c}^i_t(s^t) + \sum_{s_{t+1}} \widetilde{a}^i_{t+1}(s_{t+1}, s^t)\widetilde{Q}_t(s_{t+1}|s^t) \leq y^i_t(s^t) + \widetilde{a}^i_t(s^t).$$

- At $t = 0$, $\widetilde{a}^i_0(s_0) = \underbrace{\Upsilon^i_0(s_0)}_{=0}$ so that the BC in 0 writes

$$\widetilde{c}^i_0(s_0) + \underbrace{\sum_{s_1} \widetilde{a}^i_1(s_1, s_0)\widetilde{Q}_0(s_1|s_0)}_{\sum_{\tau=1}^{\infty}\sum_{s^t}[c^i_t(s^t)-y^i_t(s^t)]q^0_t(s^t) \text{ from } (\dagger)} = y^i_0(s_0) + \underbrace{\widetilde{a}^i_0(s_0)}_{=0}.$$

- From the intertemporal BC in the time-0 trading equilibrium,

$$y^i_0(s_0) - \sum_{\tau=1}^{\infty}\sum_{s^t}[c^i_t(s^t) - y^i_t(s^t)]q^0_t(s^t) = c^i_0(s_0) \quad \text{and therefore} \quad \widetilde{c}^i_0(s^0) = c^i_0(s^0).$$

9. Equivalence of allocations under time-zero trading and sequential trading.

- $\widetilde{c}_0^i(s^0) = c_0^i(s^0)$.
- Therefore, the proposed portfolio strategy attains the same consumption plan as in the competitive equilibrium of the Arrow-Debreu economy
- But is that the best choice for agents?
- Yes, as the natural debt limit precludes choosing a consumption plan with higher utility.