2023-2024 – Econ 0107 – Macroeconomics I

Lecture 2 : Overlapping Generations (OLG) Models

(Chapter 9 in Ljunqvist & Sargent 4th edition)

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Changes from version 1.0 are in red

1. Endowment and Preference

- Agents : $i = 0, 1, ..., +\infty$
- i is the period of birth
- Agents live for two periods
- $U^i(c^i) = u(c^i_i) + u(c^i_{i+1})$
- $U^0(c^0) = u(c^0_1)$
- Endowments $(y^i_i, y^i_{i+1}) \in \mathbb{R}^{+\star}$, $y^i_t = 0$ if $t \neq i, i+1$
- Deterministic economy
- Perishable good
- The economy starts in period 1

2. Time-0 Trading

- A special case (preferences, endowments) of the previous lecture.
- Clearing house at time 0 that posts prices and, at those prices, compiles aggregate demand and supply for goods in different periods.

Definition 1 (Equilibrium price vector)

An equilibrium price vector makes markets for all periods $t \geq 2$ *clear, but there may be excess supply in period 1*

- Excess supply in period 1 is possible because it can be given to the old without affecting equilibrium prices.
- Reason is that then old of period 1 consume all what they are given

2. Time-0 Trading

- Prices q_t^0
- Hh budget constraint (BC)

$$\sum_{t=1}^{\infty} q_t^0 c_t^i \leq \sum_{t=1}^{\infty} q_t^0 y_t^i$$

  with Lagrange multiplier μ^i.
- FOC are

$$\begin{aligned} \mu^i q_i^0 &= u'(c_i^i) \\ \mu^i q_{i+1}^0 &= u'(c_{i+1}^i) \\ c_t^i &= 0 \text{ if } t \neq i, i+1 \end{aligned}$$

- Feasibility

$$c_i^i + c_i^{i-1} \leq y_i^i + y_i^{i-1} \tag{2.1.}$$

  or equivalently

$$c_t^t + c_t^{t-1} \leq y_t^t + y_t^{t-1}$$

2. Time-0 Trading

Definition 2 (Stationary allocation)

An allocation is stationary if $c_i^i = c_y$ *and* $c_{i+1}^i = c_o$ $\forall i > 0$.

- Note that $c_1^0 = c_o$ is not required.

- Assumption: $y_i^i = 1 - \varepsilon$, $y_{i+1}^i = \varepsilon$, $y_t^i = 0$ otherwise
- $\varepsilon \in [0, 1/2]$: more endowment received when young
- Many equilibria
- Look at the two stationary ones that we will guess and verify.
- H and L equilibrium (High and Low interest rate)

2.1. Example Equilibria

H stationary equilibrium

- Guess $q_t^0 = 1 \; \forall t, c_i^i = c_{i+1}^i = 1/2, \; c_1^0 = \varepsilon$
- Check:
  - Feasible for $t > 1$
  - Feasible for $t = 1$
  - FOC is satisfied

$$\frac{u'(c_i^i)}{u'(c_{i+1}^i)} = \frac{q_i^0}{q_{i+1}^0}$$

- Notes :
  - a lot of intergenerational trade
  - some goods are wasted in period 1 (but that is an equilibrium outcome)
  - $\frac{q_{i+1}^0}{q_i^0} = \alpha_i = \frac{1}{1 + r_{i,i+1}} \rightsquigarrow \frac{q_{i+1}^0}{q_i^0} = 1$ corresponds to $r_{i,i+1} = 0 \rightsquigarrow$ High interest rate (compared to the other stationary equilibrium)

- Guess $c_i^i = y_i^i \ \forall i$, $\dfrac{q_{i+1}^0}{q_i^0} = \dfrac{u'(\varepsilon)}{u'(1-\varepsilon)} = \alpha > 1$
- Check:
  - × Feasible for $t \geq 1$
  - × Feasible for $t = 0$
  - × FOC is satisfied
- Notes :
  - × prices prevent any intergenerational trade $\rightsquigarrow$ autarky
  - × $\dfrac{q_{i+1}^0}{q_i^0} = \dfrac{1}{1 + r_{i,i+1}} > 1 \rightsquigarrow r_{i,i+1} < 0 \rightsquigarrow$ Low interest rate (compared to the other stationary equilibrium)

2.2. Relation with Welfare Theorems

- None of those two stationary equilibria are PARETO optimal
- The H equilibrium allocation is wasting some goods in period 1
- There is room to set up a giveaway program to the initial old that makes them better off and costs subsequent generations nothing.
- In H equilibrium every generation after the initial old one is better off and no generation is worse off than in L equilibrium .
- L Equilibrium is not PARETO optimal because it is dominated by H equilibrium.
- Note that H and L fail to satisfy one of the assumptions needed to deliver the first fundamental theorem of welfare economics.
- That condition is the assumption that the value of the aggregate endowment at the equilibrium prices is finite.
- If horizon was finite, equilibrium H would not exist and L would be PARETO optimal.

2.3. Non Stationary Equilibria

Definition 3 (Offer curve)

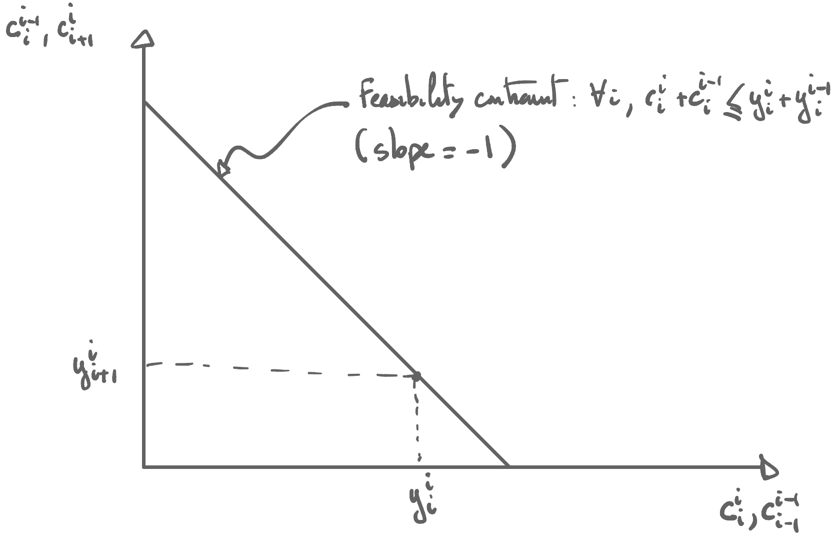
The household's offer curve is the locus of (c_i^i, c_{i+1}^i) *that solves* $\max U(c^i)$ *s.t. the BC*

$$c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i$$

for $\alpha_i \in \mathbb{R}^{+\star}$

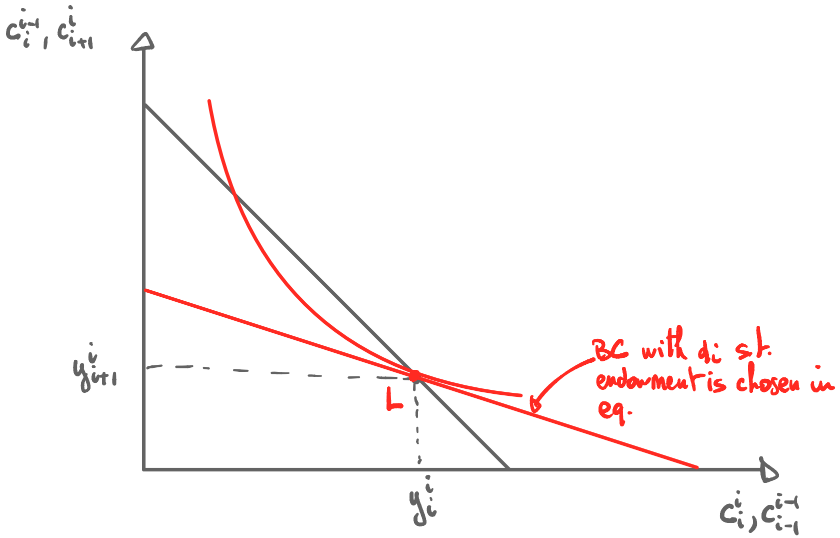
- Recall that
$$\alpha_i = \frac{q_{i+1}^0}{q_i^0} = \frac{1}{R_{i,i+1}} = \frac{1}{1 + r_{i,i+1}}$$
- The offer curve solves:
$$\begin{aligned} c_i^i + \alpha_i c_{i+1}^i &= y_i^i + \alpha_i y_{i+1}^i \\ \frac{u'(c_{i+1}^i)}{u'(c_i^i)} &= \alpha_i \end{aligned}$$

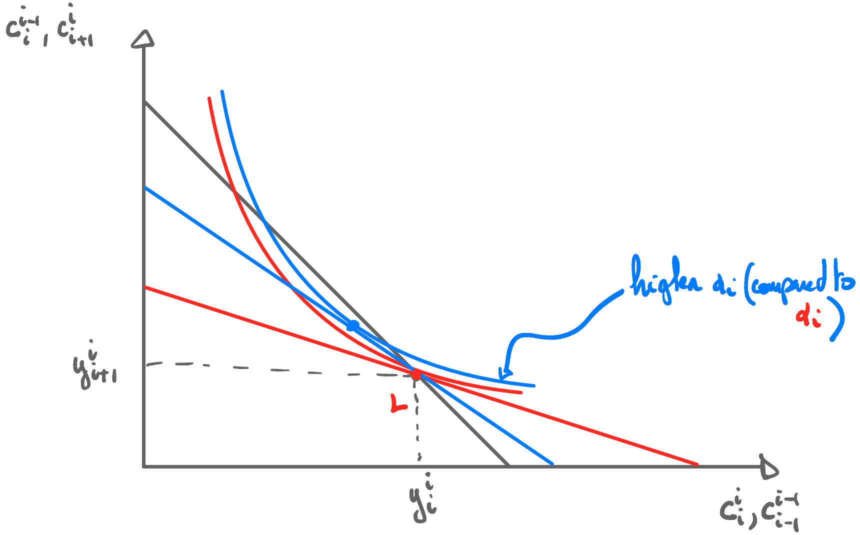
$\rightsquigarrow \psi(c_i^i, c_{i+1}^i) = 0$


c_i^{i-1}, c_{i+1}^i
Feasibility constraint: $\forall i$, $c_i^i + c_i^{i-1} \leq y_i^i + y_i^{i-1}$
(slope = -1)
y_{i+1}^i
y_i^i
c_i^i, c_{i-1}^{i-1}

2.3. Non Stationary Equilibria

The Offer Curve

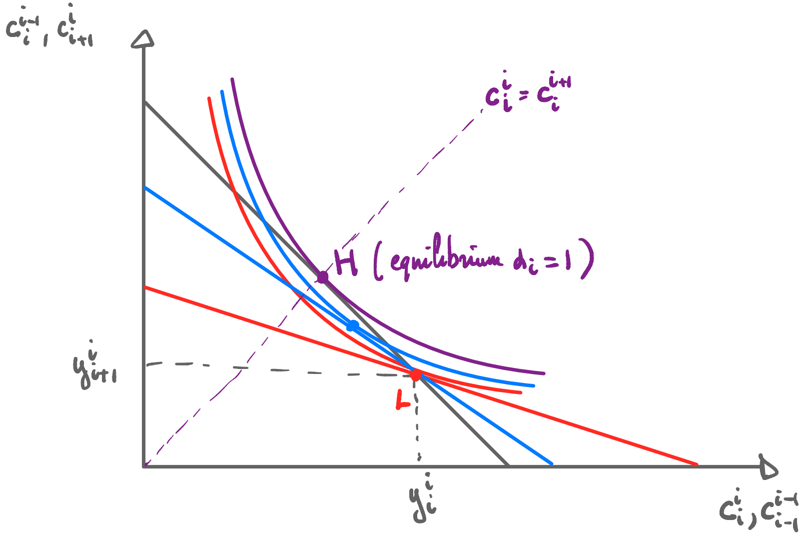




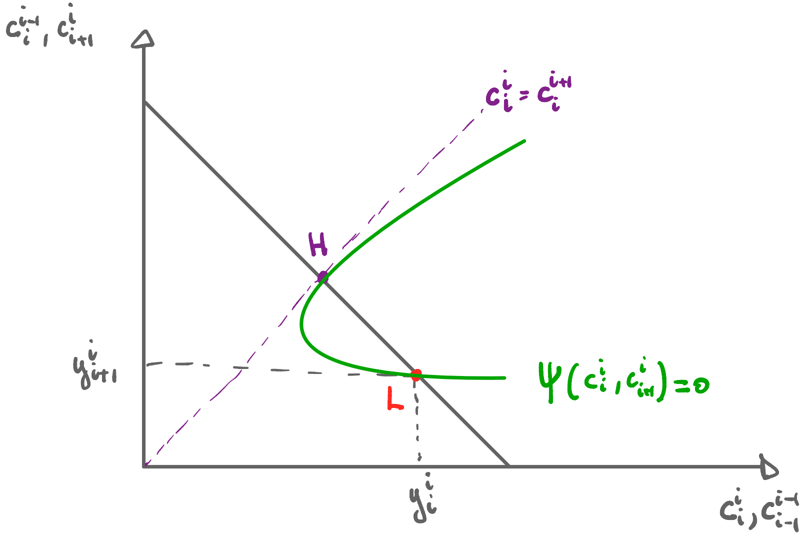
c_i^{i-1}, c_{i+1}^i
y_{i+1}^i
y_i^i
c_i^i, c_{i-1}^{i-1}
higher d_i (compared to d_i)

2.3. Non Stationary Equilibria

The Offer Curve

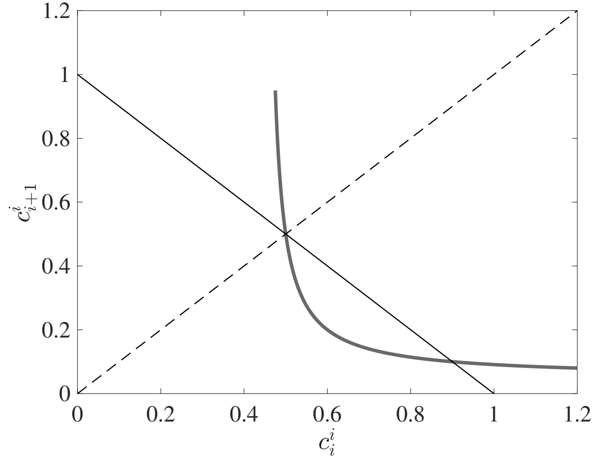


The Offer Curve



2.3. Non Stationary Equilibria

The Offer Curve with $u = \log$ and $\varepsilon = .1$



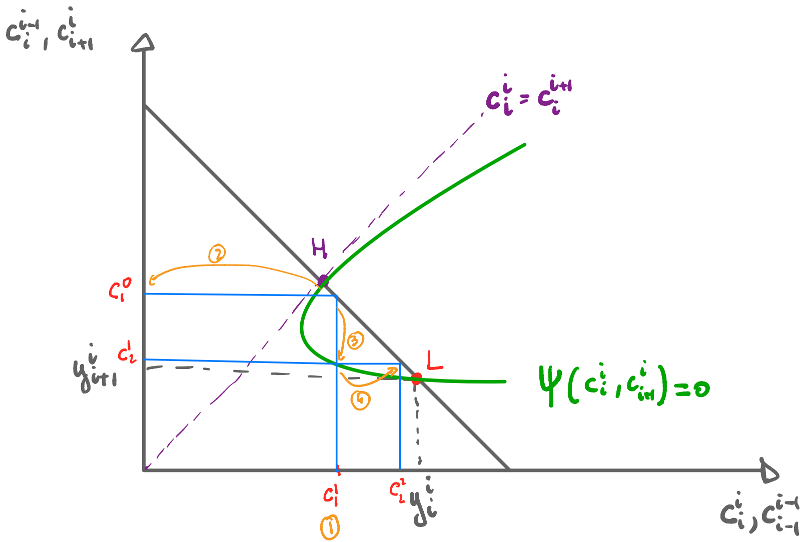
- One can construct a non-stationary equilibrium using the offer curve, i.e. using the recursion:

$$\begin{aligned} \psi(c_i^i, c_{i+1}^i) &= 0 \\ c_i^i + c_i^{i-1} &= y_i^i + y_i^{i-1} \end{aligned}$$

1. Choose c_1^1 in $[c_y^H, c_y^L]$ (arbitrarily)
2. Use feasibility to find the maximum c_1^0 possible
3. Use offer curve to find c_2^1 (and α_1) s.t. (c_1^1, c_2^1) maximises U^1 when prices are α_1.
4. From c_2^1, use feasibility to find c_2^2
5. then repeat steps 3 to 4.

- Note that allocations converge towards L

Example 1



c_i^{i-1}, c_{i+1}^i
$c_i^i = c_i^{i+1}$
H
L
c_1^0
c_2^1
y_{i+1}^i
$\Psi(c_i^i, c_{i+1}^i) = 0$
c_1^1
c_2^2
y_i^i
c_i^i, c_{i-1}^{i-1}

- Assume that the initial old has $y_1^0 = \varepsilon$ in period 0 and "$y_\infty^0 = \delta = 1 - \varepsilon$ in period is $+\infty$"
- More formally, wealth of the initial old is

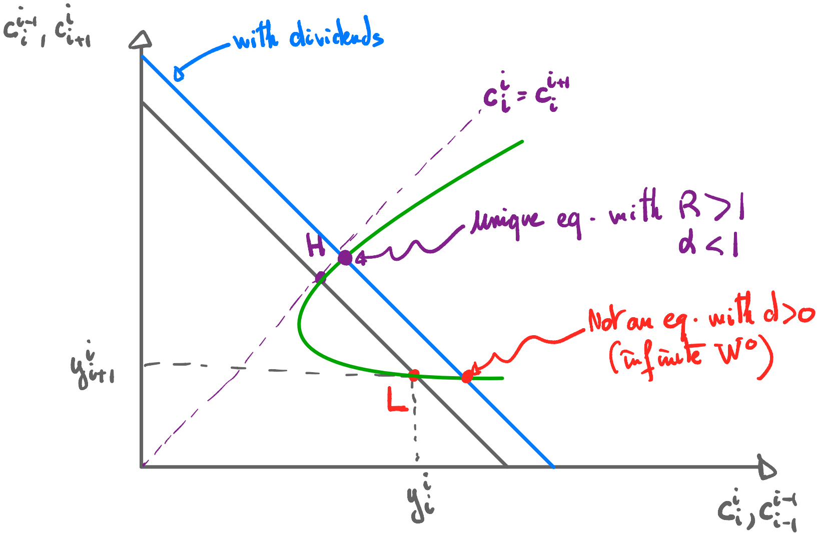
$$\begin{aligned} q_1^0 W^0 &= q_1^0 y_1^0 + \lim_{t \to \infty} q_t^0 y_t^0 \\ &= q_1^0 \varepsilon + \delta \lim_{t \to \infty} q_t^0 \end{aligned}$$

- At the L equilibrium, $\dfrac{q_t^0}{q_{t-1}^0} = \alpha > 1 \leadsto q_t^0 = \alpha^t q_1^0$
- Therore, $W^0 = \varepsilon + \delta \lim\limits_{t \to \infty} \alpha^t \to +\infty$
- The initial old has an infinite wealth in period 1 $\leadsto$ will demand infinite consumption $\leadsto$ not an equilibrium.
- There is therefore only the H stationary equilibrium

- Assume that the initial old has a Lucas tree, that pays d each period.
- BC of the initial old : $q_1^0 c_1^0 = d \sum_{t=1}^{\infty} q_t^0 + q_1^0 y_1^0$
- Same offer curve, but the feasibility condition is shifted up by d
- With a Lucas tree, the only stationary equilibria is H (infinite wealth of the initial old at L because the interest rate is too low.
- $\alpha < 1$ $(R > 1)$ at the H stationary eq.
- Note also that we can rule out all the non-stationary candidates as they converge to L
- The only equilibrium is the stationary equilibrium H.

2.4. Computing Equilibria

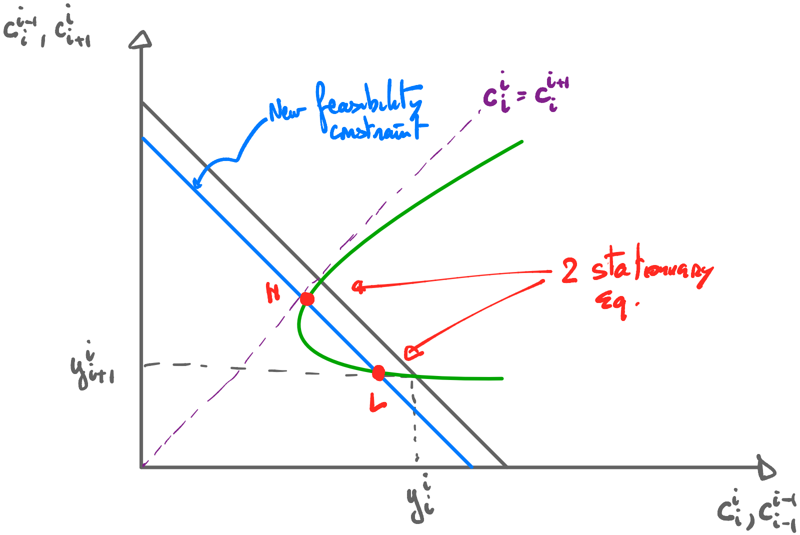
Example 3: Lucas tree



- Feasibility becomes
$$c_i^i + c_i^{i-1} + g = y_i^i + y_i^{i-1}$$
- Now there are two stationary equilibria, with both low interest rate $(\alpha > 1)$
- Low interest rate equilibria cannot be ruled out as previously.

2.4. Computing Equilibria

Example 4: Government expenditures



- Offer curve:

$$
\begin{aligned}
c_i^i + \alpha_i c_{i+1}^i &= y_i^i + \alpha_i y_{i+1}^i = 1 - \varepsilon + \alpha_i \varepsilon \\
\frac{c_i^i}{c_{i+1}^i} &= \alpha_i
\end{aligned}
$$

which gives

$$
\begin{aligned}
c_i^i &= \frac{1}{2}\left(1 - \varepsilon + \alpha_i \varepsilon\right) \\
c_{i+1}^i &= \frac{c_i^i}{\alpha_i}
\end{aligned}
$$

- Plug in feasibility, which writes

$$c_i^i + c_i^{i-1} = 1$$

to obtain the equilibrium price recursion

$$\alpha_i = \frac{1}{\varepsilon} - \frac{\frac{1}{\varepsilon} - 1}{\alpha_{i-1}}$$

- We have two stat. eq. $\alpha = 1$ and $\alpha = \dfrac{1-\varepsilon}{\varepsilon} > 1$ because $\varepsilon < .5$

3. Sequential Trading

- Now trade takes place every period
- No IOUs, as agents of the same generation are identical, and agents of two different generations do not meet in two consecutive periods.
- We add a durable asset (fiat money, gov bonds, Lucas tree)

4. Money

- Based on Samuelson [1958]
- Same model than before, but in $t = 1$, old are endowed with $M > 0$ units of intrinsically worthless currency.
- P_t is the price on 1 u of good in term of the currency
- $1/P_t$ is the price of money (in term of good)
- From $i \geq 1$ onwards, the young buys m_i^i units of money from the old
- The old sells the currency to the young against goods

- BC of a young born in $i \geq 1$:

$$
\begin{aligned}
c_i^i + \frac{m_i^i}{P_i} &\leq y_i^i \\
c_{i+1}^i &\leq \frac{m_i^i}{P_{i+1}} + y_{i+1}^i \\
m_i^i \geq 0 &
\end{aligned}
$$

- If $m_i^i \geq 0$, the we have an intertemporal BC

$$
c_i^i + c_{i+1}^i \left(\frac{P_{i+1}}{P_i} \right) \leq y_i^i + y_{i+1}^i \left(\frac{P_{i+1}}{P_i} \right) \tag{4.4}
$$

- Note that with $\dfrac{P_{i+1}}{P_i} = \alpha_i = \dfrac{q_{i+1}^0}{q_i^0}$, we have the same IBC than (2.1.) (in the date-0 trading model)

Definition 4 (Nominal price sequence)

A nominal price sequence is a positive sequence $\{P_i\}_{i \geq 1}$

Definition 5 (Monetary equilibrium)

An equilibrium with valued fiat money (monetary equilibrium) is a feasible allocation and a nominal price sequence with $P_i < \infty$ *for all* i

- Remark: if $P_t \to \infty$, then $1/P_t \to 0$: the price of money is zero, i.e. money is not valued $\rightsquigarrow$ non-monetary equilibrium (autarky)

4.1. Computing more equilibria with valued fiat money

- Household optimal decision is sumarized by

$$y_i^i - c_i^i = s(\alpha_i, y_i^i, y_{i+1}^i)$$

- Equilibrium condition is

$$\underbrace{\frac{M}{P_i}}_{\text{real dissaving of gen. } i-1} = \underbrace{s(\alpha_i, y_i^i, y_{i+1}^i)}_{\text{real saving of gen. } i}$$

with

$$\alpha_i = \frac{P_{i+1}}{P_i}$$

$\rightsquigarrow$ we have a difference equation in P_i, P_{i+1} that we need to solve for $\{P_i\}_{i=1}^{\infty}$

4.1. Computing more equilibria with valued fiat money

Example: $u = \log c$, endowments (ω_1, ω_2), $\omega_1 > \omega_2$

- $\max \log c_i^i + \log c_{i+1}^i$
  s.t. $c_i^i + \alpha_i c_{i+1}^i \leq \omega_1 + \alpha_i \omega_2 \quad (\lambda^i)$
- FOC: $\dfrac{1}{c_i^i} = \lambda^i$ and $\dfrac{1}{c_{i+1}^i} = \alpha_i \lambda^i \Rightarrow c_i^i = \alpha_i c_{i+1}^i$
- Plug in the BC: $2c_i^i = y_i^i + \alpha_i y_{i+1}^i$
- Compute savings: $s(\alpha_i, \omega_1, \omega_2) = y_i^i - c_i^i = \dfrac{1}{2}(\omega_1 - \underbrace{\alpha_i}_{\frac{P_{i+1}}{P_i}} \omega_2)$
- Equilibrium: $\dfrac{M}{P_i} = \dfrac{1}{2}(\omega_1 - \alpha_i \omega_2) \Rightarrow$

$$P_i = \frac{2M}{\omega_1} + \underbrace{\frac{\omega_2}{\omega_1}}_{<1} P_{i+1}$$

4.1. Computing more equilibria with valued fiat money

Example: $u = \log c$, endowments (ω_1, ω_2), $\omega_1 > \omega_2$

$$P_i = \frac{2M}{\omega_1} + \underbrace{\frac{\omega_2}{\omega_1}}_{<1} P_{i+1}$$

- Solve forward:

$$\begin{aligned} P_i &= \frac{2M}{\omega_1} + \frac{\omega_2}{\omega_1}\left(\frac{2M}{\omega_1} + \frac{\omega_2}{\omega_1} P_{i+2}\right) \\ &= \cdots \\ &= \frac{2M/\omega_1}{1-\omega_2/\omega_1} + \underbrace{\lim_{T+\infty} \left(\frac{\omega_2}{\omega_1}\right)^T P_T}_{0 \text{ at stationary monetary equilibrium}} \end{aligned}$$

- Therefore in stationary equilibrium $P_i = \dfrac{2M}{\omega_1 - \omega_2}$

4.2 Equivalence of equilibria

- Let's be in the case where endowments are $(1-\varepsilon, \varepsilon)$, $\varepsilon < 1/2$

Proposition 1 (Time-0 trading and sequential trading)

Let $\overline{c}^i$ denote a competitive equilibrium with time-0 trading, and suppose it satisfies $\overline{c}^i_i < y^i_i$ (positive savings), then $\exists$ an equilibrium with sequential trading of the monetary economy with $c^i_i = \overline{c}^i_i$, $c^i_{i+1} = \overline{c}^i_{i+1}$ $\forall i \geq 1$

- Compute $\alpha_i = \dfrac{q^0_{i+1}}{q^0_i}$
- Set $m^i_i = M$
- Derive P_1 from

$$\frac{M}{P_1} = \underbrace{y^1_1 - \overline{c}^1_1}_{\text{if positive, then } P_1 > 0 \text{ and unique}}$$

- (note that only $\dfrac{M}{P_1}$ matters, not M and P_1 separately)
- Construct $\{P_i\}_{i=1}^{\infty}$ using $P_{i+1} = \alpha_i P_i$
- Allocate to period-0 old:

$$c^0_1 = y^0_1 + \frac{M}{P_1} = \underbrace{y^0_1}_{\varepsilon} + \underbrace{y^1_1 - \overline{c}^1_1}_{<1-\varepsilon}$$

- *QED*

4.2 Equivalence of equilibria

Proposition 2 (Sequential trading and time-0 trading)

Let $\overline{c}^i$ be an equilibrium for the sequential trading monetary economy. There is a time-0 trading economy with the same allocations provided that some transfers are made to the old of period 1

- Proof: Do transfers such that

$$c_1^0 = y_1^0 + \underbrace{(y_1^1 - \overline{c}_1^1)}_{\text{transfers}}$$

- Construct $\frac{q_{i+1}^0}{q_i^0} = \alpha_i = \frac{P_{i+1}}{P_1} \rightsquigarrow$ with these prices q^0, $c^i = \overline{c}^i$ is a time-0 trading equilibrium.

5. Deficit finance

- Assume sequential trading, N agents
- $(y_i^i, y_{i+1}^i) = (\omega_1, \omega_2)$, $\omega_1 > \omega_2$
- Taxes (τ_1, τ_2)
- Government:

$$M_t - M_{t-1} = P_t \underbrace{(g - \tau_1 - \tau_2)}_{\text{deficit } d} \qquad (\star)$$

- Note: if "$P_t = +\infty$" (non monetary equilibrium), then $g = \tau_1 + \tau_2$

- for generations $i \geq 1$:

$$\max \quad u(\omega_1 - \tau_1 - s) + u(\omega_2 - \tau_2 + R_t s)$$

- with $R_t = \dfrac{P_t}{P_{t+1}} \rightsquigarrow$ solution: $s_t = f(R_t)$

Definition 6 (Equilibrium with valued fiat money)

An equilibrium with valued fiat money is a pair of sequences $\{M_t, P_t\}$ such that

1. *given* $\{P_t\}$, $\dfrac{M_t}{P_t} = f(R_t)$,
2. $R_t = P_t / P_{t+1}$,
3. *The gvt. BC is satisfied.*

$$f(R_t) = \frac{M_t}{P_t} \Longleftrightarrow f(R_t) = \frac{M_{t-1}}{P_t} + \frac{M_t - M_{t-1}}{P_t}$$

- savings of the young
- dissavings of the old
- deficit $d = g - \tau_1 - \tau_2$ (dissaving of the gvt) (real value of currency printing)

- Gvt. BC:

$$\begin{cases} \frac{M_t}{P_t} = \frac{M_{t-1}}{P_{t-1}} \times \frac{P_{t-1}}{P_t} + d \quad \forall t \geq 2 \\ \frac{M_1}{P_1} = \frac{M_0}{P_1} + d \end{cases}$$

- Gvt. BC:

$$\begin{cases} \dfrac{M_t}{P_t} = \dfrac{M_{t-1}}{P_{t-1}} \times \dfrac{P_{t-1}}{P_t} + d \;\; \forall t \geq 2 \\ \\ \dfrac{M_1}{P_1} = \dfrac{M_0}{P_1} + d \end{cases}$$

- Using $\dfrac{M_t}{P_t} = f(R_t)$:

$$\begin{cases} f(R_t) = f(R_{t-1}) \times R_{t-1} + d \;\; \forall t \geq 2 \\ \\ f(R_1) = \dfrac{M_0}{P_1} + d \end{cases}$$

$$\begin{cases} f(R_t) = f(R_{t-1}) \times R_{t-1} + d \quad \forall t \geq 2 \\ \\ f(R_1) = \dfrac{M_0}{P_1} + d \end{cases}$$

- This is a difference equation in R_t that we can solve for a given $\dfrac{M_0}{P_1}$
- $\dfrac{M_0}{P_1} =$ "how much is given to the period 1 old"
- Note: Only $\dfrac{M_0}{P_1}$ matters (not M_0 and P_1 separately)

- Steady state:

$$f(R) = f(R) \times R + d \iff \underbrace{f(R)}_{\frac{M_t}{P_t}} \times \underbrace{(1-R)}_{\substack{\text{“tax rate”} \\ \text{on real balances}}} = \underbrace{d}_{\text{deficit}}$$

$$f(R) = \frac{M_0}{P_1} + d$$

- pins down R
- pins down P_1

$$f(R) = f(R) \times R + d \Longleftrightarrow$$

$$\underbrace{f(R)}_{\frac{M_t}{P_t}} \quad \times \quad \underbrace{(1-R)}_{\substack{\text{“tax rate”} \\ \text{on real balances}}} \quad = d$$

and

$$f(R) = \frac{M_0}{P_1} + d$$

- We have

$$\frac{M_t}{P_t}(1-R) = d$$

- Note that

$$R_t = \frac{P_t}{P_{t+1}} = \frac{1}{1+\pi_{t+1}}$$

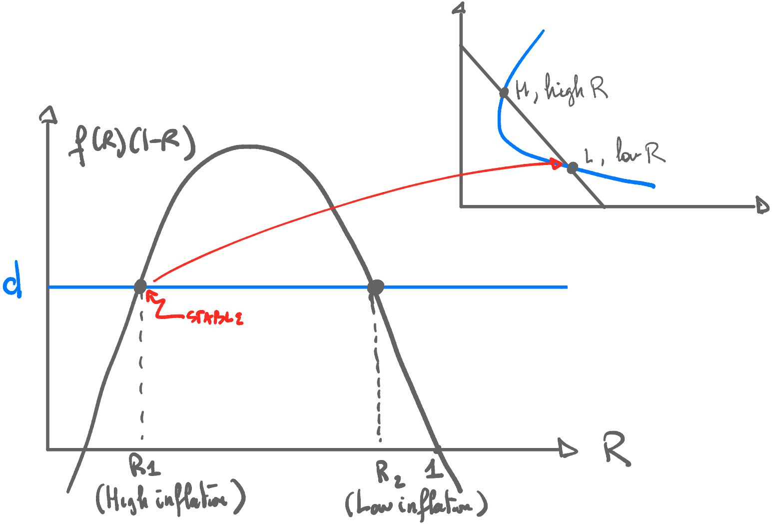
whith π_{t+1} is the inflation rate.

- Inflation tax:

$$\underbrace{1-R_t}_{\text{tax rate on } \frac{M_t}{P_t}} = 1 - \frac{1}{1+\pi_{t+1}} = \frac{\pi_{t+1}}{1+\pi_{t+1}} \approx \underbrace{\pi_{t+1}}_{\text{inflation rate}}$$

5.1. Stationary state and the LAFFER curve

With $u(c) = \log(c)$, $f(R_t) = \dfrac{\omega_1 - \tau_1}{2} - \dfrac{\omega_2 - \tau_2}{2R_t}$



5.1. Stationary state and the LAFFER curve

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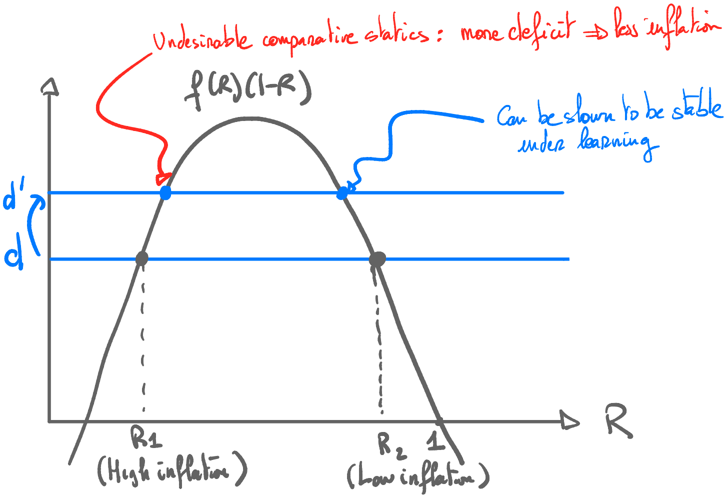

6. Equivalent setups

- Take a model with gvt. deficit (No taxes + $\{g_t\}$)
- There exist three equivalent structures:
  1. sequential trading + fiat currency
  2. sequential trading + gvt. indexed bonds
  3. time-0 trading with Arrow-Debreu securities

Definition 7 (Sequential trading + fiat currency equilibrium)

An equilibrium is a sequence $\{M_t, P_t\}_{t=1}^{+\infty}$ *with* $0 < P_t < +\infty$, $M_t > 0$ *such that*

1. *given* $\{P_t\}$, $\{M_t\}$ *satisfies*

$$M_t = Argmax_{\widetilde{M}} \quad u\left(y_t^t - \frac{\widetilde{M}}{P_t}\right) + u\left(y_{t+1}^t + \frac{\widetilde{M}}{P_{t+1}}\right)$$

2. *Gvt. BC holds for* M_0 *given*

$$M_t - M_{t-1} = P_t g_t$$

- No money
- We introduce bonds
- B_t: sold by the gvt. to young of period t (1 unit of bond for $\frac{1}{R_t}$ units of good in t, each unit of bond pays 1 unit of good in $t+1$).
- B_1: endowment of the old of period 1, pays 1 unit of good per unit of bond in period 1

Definition 8 (Sequential trading + gvt. indexed bonds equilibrium)

An equilibrium with bonds financed government deficits is a sequence $\{B_{t+1}, R_t\}_{t=1}^{+\infty}$ *such that*

1. *given* $\{R_t\}$, $\{B_{t+1}\}$ *satisfies*

$$B_{t+1} = Argmax_{\widetilde{B}} \quad u\left(y_t^t - \frac{\widetilde{B}}{R_t}\right) + u\left(y_{t+1}^t + \widetilde{B}\right)$$

2. *Gvt. BC holds for* B_1 *given*

$$\frac{B_{t+1}}{R_t} = B_t + g_t$$

Proposition 3 (Equivalence)

The two equilibria 6.1. and 6.2. are isomorphic.

- Proof:
  - Take equilibrium 6.1. and define $B_t = \dfrac{M_{t-1}}{P_t}$ and $R_t = \dfrac{P_t}{P_{t+1}}$.
  - With these B and R, the consumptions of equilibrium 6.1. are also equilibrium consumptions of 6.2.
  - The gvt. BC is satisfied in equilibrium 6.2.

- The same allocations than 6.1. and 6.2. can be obtained in equilibrium 6.3. if we transfer the right amount of goods to the old of period 1.
- Let B_1^g be claims to time 1 consumption owed by the gvt. to the old of time 1.

Definition 9 (Time-0 trading with Arrow-Debreu securities equilibrium)

An equilibrium with time-0 trading is a B_1^g, a price system $\{q_t^0\}_{t=1}^{+\infty}$ and savings $\{s_t\}_{t=1}^{+\infty}$ such that

1. *given $\{q_t\}$, $\{s_t\}$ satisfies*

$$s_t = Argmax_{\tilde{s}} \quad u\left(y_t^t - \tilde{s}_t\right) + u\left(y_{t+1}^t + \frac{q_t^0}{q_{t+1}^0}\tilde{s}_t\right)$$

2. *Gvt. intertemporal BC holds:*

$$\underbrace{q_1^0 B_1^g}_{negative} + \underbrace{\sum_{t=1}^{+\infty} q_t^0 g_t}_{positive} = 0$$

Note that $q_1^0 B_1^g < 0$ represents negative net worth for the houshold.

- In that time-0 trading equilibrium, one can construct a sequence of public debt using

$$q_{t+1}^0 B_{t+1}^g = q_t^0 B_t^g + q_t^0 g_t \qquad \forall t \geq 1$$

- B_1^g can be obtained from the gvt. intertemporal BC:

$$\begin{aligned} q_1^0 B_1^g &= -q_1^0 g_1 + q_2^0 B_2^g \\ & \quad -q_1^0 g_1 + \left(-q_2^0 g_2 + q_3^0 B_3^g\right) \\ & \quad \dots \\ & \quad -\sum_{t=1}^{+\infty} q_t^0 g_t + \underbrace{\lim_{T \to +\infty} q_{t+T}^0 B_{t+T}^g}_{\text{impose } = 0} \end{aligned}$$

- Assume $N_{t+1} = nN_t$, $n > 0$
- Consider the equilibrium with money-funded deficit
- M_t = per capita level of currency, g = per capita gvt. expenditures
- Money supply = $N_t M_t$
- Gvt. BC : $N_t M_t - N_{t-1} M_{t-1} = N_t P_t g$
- Divide by $N_t P_t$:

$$\frac{M_t}{P_{t+1}} \frac{P_{t+1}}{P_t} - \frac{N_{t-1}}{N_t} \frac{M_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} = g$$

or equivalently

$$\frac{M_t}{P_{t+1}} \frac{P_{t+1}}{P_t} = n^{-1} \frac{M_{t-1}}{P_t} + g$$

or

$$M_t - n^{-1} M_{t-1} = P_t g \qquad \rightsquigarrow \text{same as before but for } n^{-1}$$

- Sequential trading, no gvt.
- $N_t = nN_{t-1}$
- endowments (y_1, y_2)
- $u(c_t^t, c_{t+1}^t)$
- $\theta(c_1, c_2) = \dfrac{u_1(c_1, c_2)}{u_2(c_1, c_2)}$ Marginal Rate of Substitution
- Assume θ is well behaved:
  - × $\theta \to 0$ when $\dfrac{c_2}{c_1} \to 0$
  - × $\theta \to +\infty$ when $\dfrac{c_2}{c_1} \to +\infty$

- $M_0 = N_0 m_0^0$
- for $t \geq 1$, $M_t = zM_{t-1}$, $z > 0$: transfer or tax $(z-1)M_{t-1}$ that is equally distributed to the old of period t in a lump sum way
- BCs of a generation t agent:

$$
\begin{aligned}
c_t^t + \frac{m_t^t}{P_t} &\leq y_1 \\
c_{t+1}^t &\leq y_2 + \frac{m_t^t}{P_{t+1}} + \frac{(z-1)}{N_t}\frac{M_t}{P_t} \\
m_t^t &\geq 0
\end{aligned}
$$

- Non monetary equilibrium (autarky):

$$\theta_{\text{aut}} = \frac{u_1(y_1, y_2)}{u_2(y_1, y_2)}$$

- Two questions
  1. Under what circumstances does a monetary equilibrium exists?
  2. When it exists, under what circumstances does it improve welfare as compared to the non monetary equilibrium?

Preview: when $z = n = 1$ and $u = u(c_1) + u(c_2)$

Proposition 4 (Existence)

$\theta_{aut} < 1$ is N and S for the existence of at least one monetary equilibrium

- Idea of the proof:
  1. $\theta_{\text{aut}} = \dfrac{u'(y_1)}{u'(y_2)}$
  2. $\theta_{\text{aut}} < 1$ implies $y_1 > y_2$ $\rightsquigarrow$ "desire to save" $\rightsquigarrow$ "demand for asset" $\rightsquigarrow$ Money will be positively valued

7. Optimality and existence of monetary equilibria

When $z = n = 1$ and $u = u(c_1) + u(c_2)$

Proposition 5 (Optimality)

$\theta_{aut} \geq 1$ is N and S for the optimality of the non-monetary equilibrium

- Idea of the proof (by contradiction):
  - × Assume $\theta_{\text{aut}} < 1$. This implies $y_1 > y_2$ $\rightsquigarrow$ autarky is not PARETO optimal

Preview: when $z = n = 1$ and $u = u(c_1) + u(c_2)$

- Summary: if $y_1 > y_2$,
  - × Proposition 5: Non-monetary eq. is not efficient
  - × Proposition 4: (at leat one) Monetary eq. exists

$\rightsquigarrow$ It can be generalized for any z and n positive.

Proposition 6 (Existence of a monetary equilibrium)

$\theta_{aut} \times z < n$ ("the interest rate is low in autarky") is N and S for existence of at least one monetary equilibrium.

Proposition 7 (Optimality)

$\theta_{aut} > n$ is N and S for the optimality of autarky

7.1. Balasko-Shell criterion for optimality

- Make assumptions on endowments and preferences to rule out pathological cases

Proposition 8 (Balasko-Shell (1980) criterion)

An allocation is Pareto *optimal if*

$$\sum_{t=1}^{\infty} \prod_{s=1}^{t} (1 + r_s) = +\infty$$

- In words, the real interest rate should not be too low for optimality of equilibria