2023-2024 – Econ 0107 – Macroeconomics I

Lecture 4 : Fiscal Policies in a Growth Model

(Chapter 11 in LJUNQVIST & SARGENT)

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Changes from version 1.0 are in red

1. Introduction

- Complete market economy
- Time-0 trading
- Add production and taxes

- No uncertainty
- Representative household (hh)

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t) \tag{2.1}$$

- Typically, in DSGEs:
  - $U = u(c) + v(1-n)$
  - $U = \log c + \zeta \log(1-n)$
  - $U = \log c + \zeta \times (1-n)$
  - $U = u(c)$ (fixed labor supply)

- Technology:

$$F(k_t, n_t) \geq g_t + c_t + x_t \qquad (2.2.a)$$

$$k_{t+1} = (1-\delta)k_t + x_t \qquad (2.2.b)$$

$\rightsquigarrow$

$$g_t + c_t + k_{t+1} \leq F(k_t, n_t) + (1-\delta)k_t \qquad (2.3)$$

- F is a neoclassical production function: linearly homogenous of degree 1: $F(\lambda k, \lambda n) = \lambda^1 F(k, n)$
- Euler theorem: $F_k k + F_n n = \underbrace{\lambda}_{1} F$
- Example: $F = k^\alpha n^{1-\alpha}$, $0 < \alpha < 1$

2.2. Components of a competitive equilibrium

- (Representative) Hh: owns capital, makes investment decisions, sells labour and capital services to the representative firm
- (Representative) Firm: rents labour and capital to produce final good
- price system $\{q_t, \eta_t, w_t\}$:
  - × pre-tax prices
  - × q_t (formerly denoted q_t^0): price of one unit of investment or consumption in t in units of time 0 numéraire.
  - × η_t: price of capital services in units of time t good
  - × w_t: price of labour services in units of time t good

Definition 1

A govt expenditure and tax plan that satisfies the govt budget constraint is budget-feasible

- Competitive equilibria are indexed by alternative budget-feasible govt policies
- Hh budget constraint:

$$\sum_{t=0}^{\infty} q_t \left((1+\tau_{ct})c_t + (k_{t+1} - (1-\delta)k_t)\right) \leq \sum_{t=0}^{\infty} q_t \left(\underbrace{\eta_t k_t - \tau_{kt}(\eta_t - \delta)k_t}_{(1-\tau_{kt})\eta_t k_t + \tau_{kt}\delta k_t} + (1-\tau_{nt})w_t n_t - \tau_{ht} \right) \tag{2.4}$$

- Note: depreciation allowance δk_t from gross rentals on capital.

- Govt budget constraint:

$$\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} q_t \left(\tau_{ct} c_t + \tau_{kt} (\eta_t - \delta) k_t + \tau_{nt} w_t n_t + \tau_{ht} \right) \tag{2.5}$$

- Note: if the govt was optimising, it would use only lump sum taxe τ_h.

3. Term structure of interest rates

- $\{q_t\}_{t=0}^{\infty}$ encodes the term structure of interest rates

$$q_t = q_0 \frac{q_1}{q_0} \frac{q_2}{q_1} \cdots \frac{q_t}{q_{t-1}} = q_0 m_{0,1} m_{1,2} \cdots m_{t-1,t}$$

- $m_{t,t+1} = \frac{q_{t+1}}{q_t}$ is the one-period discount factor between t and $t+1$

$$m_{t,t+1} = R_{t,t+1}^{-1} = \frac{1}{1 + r_{t,t+1}} \approx e^{-r_{t,t+1}}$$

- We can write

$$\begin{aligned} q_t &= q_0 e^{-r_{0,1}} e^{-r_{1,2}} \cdots e^{-r_{t,t+1}} \\ &= q_0 e^{-(r_{0,1}+r_{1,2}+\cdots+r_{t-1,t})} \\ &= q_0 e^{-t r_{0,t}} \end{aligned}$$

with

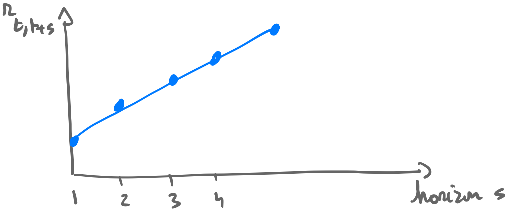
$$r_{0,t} = \frac{1}{t}(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t})$$

3. Term structure of interest rates

- $q_t = q_0 e^{-t r_{0,t}}$
- $r_{0,t}$ is the net t-period rate of interest between 0 and t.
- It is the yield to maturity on q zero coupon bon=d that matures at t.
- More generally, one can write

$$r_{t,t+s} = \frac{1}{s}(r_{t,t+1} + r_{t+1,t+2} + \cdots + r_{t+s-1,t+s})$$

- From $s = 1, 2, \ldots$, we obtain the yield curve



Detour: Interpreting the slope of the yield curve

- Take the simple endowment economy

$$\max \sum_t \beta^t \log c_t \quad \text{s.t.} \quad \sum_t q_t c_t \leq \sum_t q_t y_t \quad (\lambda)$$

- First order condition (*foc*) is $\beta^t \frac{1}{c_t} = \lambda q_t$
- Ratio of *foc* in $t+s$ and t:

$$\beta^s \frac{c_t}{c_{t+s}} = \frac{q_{t+s}}{q_t} = \frac{q_0 e^{-(t+s) r_{0,t+s}}}{q_0 e^{-(t) r_{0,t}}} = e^{-s r_{t,t+s}}$$

- Take the log and rearrange:

$$r_{t,t+s} = \gamma_{c_{t,t+s}} + \log \beta$$

  where $\gamma_{c_{t,t+s}}$ is the average per period growth rate of consumption between t and $t+s$
- Expecting lower growth in the future implies that $r_{t,t+s}$ decreases with s ("inversion of the yield curve is a predictor of recession)"

Detour: Interpreting the slope of the yield curve

FORBES > MONEY > INVESTING

Yield Curve Less Inverted, But Recession Warning Remains

Simon Moore Senior Contributor

I show you how to save and invest.

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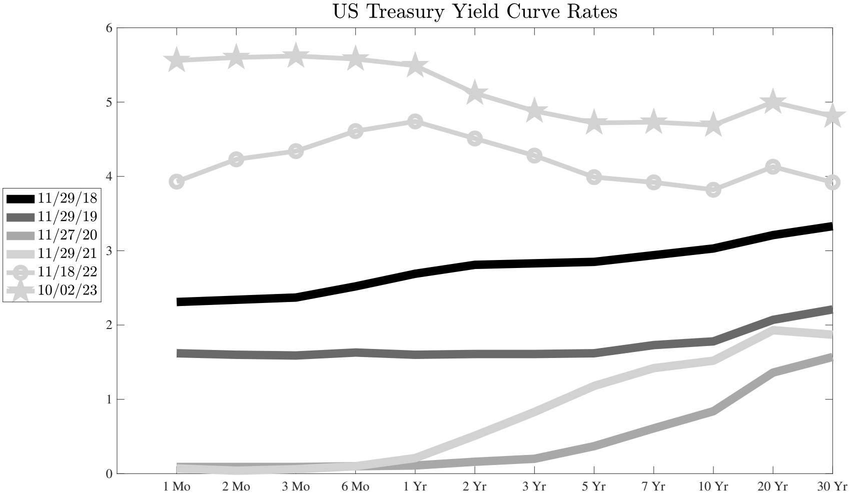
Oct 9, 2023, 10:58am EDT



US Treasury Yield Curve Rates
0
1
2
3
4
5
6
1 Mo
2 Mo
3 Mo
6 Mo
1 Yr
2 Yr
3 Yr
5 Yr
7 Yr
10 Yr
20 Yr
30 Yr
11/29/18
11/29/19
11/27/20
11/29/21
11/18/22
10/02/23

4. Sequential version of the govt budget constraint

- It is useful to describe the sequence of on-period public debt associated with the expenditures and tax revenues (but it is not needed to compute the equilibrium)
- Assume no govt debt when entering period 0.
- Let T_t be the total tax revenues:

$$T_t = \tau_{ct} c_t + \tau_{kt}(\eta_t - \delta) k_t + \tau_{nt} w_t n_t + \tau_{ht}$$

The govt intertemporal budget constraint is

$$\sum_{t=0}^{\infty} q_t (g_t - T_t) = 0 \tag{4.1}$$

that can be rewritten as

$$\underbrace{g_0 - T_0}_{\text{current deficit}} = \underbrace{\sum_{t=1}^{\infty} \frac{q_t}{q_0} (T_t - g_t)}_{\text{discounted sum of future surpluses}} \qquad (\star)$$

- in a sequential world, one can think of period 0 deficit as being financed by debt B_0:

$$B_0 = g_0 - T_0$$

- Therefore $(\star)$ implies

$$B_0 = \sum_{t=1}^{\infty} \frac{q_t}{q_0}(T_t - g_t)$$

or

$$\underbrace{\frac{q_0}{q_1}}_{R_{0,1}} B_0 = T_1 - g_1 + \underbrace{\sum_{t=2}^{\infty} \frac{q_t}{q_1}(T_t - g_t)}_{B_1}$$

or equivalently

$$g_1 + R_{0,1} B_0 = T_1 + B_1$$

- In period t, we will have

$$g_t + R_{t-1,t} B_{t-1} = T_t + B_t$$

or

$$\underbrace{B_t - B_{t-1}}_{\text{new debt issuance}} = \underbrace{g_t - T_t}_{\text{primary deficit}} + \underbrace{r_{t-1,t} B_{t-1}}_{\text{net interest payments}} \tag{4.4}$$

- The Arrow-Debreu budget constraint (4.1) ensures the no-Ponzi scheme (transversality) condition

$$\lim_{t \to \infty} q_t B_{t+1} = 0$$

4. Sequential version of the govt budget constraint

- Note:
  - × There is no loss of generality in considering only one-period debt
  - × The maturity structure of govt debt is irrelevant.

- Hh chooses $\{c_t, n_t, k_{t+1}\}$ for $t = 0, \dots$ to max U s.t. the budget constraint
- Firm chooses $\{k_t, n_t\}$ for $t = 0, \dots$ to max firm value V_0

$$V_0 = \sum_{t=0}^{\infty} q_t \underbrace{\left(F(k_t, n_t) - \eta_t k_t - w_t n_t\right)}_{\text{profit of period } t}$$

- A budget-feasible policy is an expenditure plan $\{g_t\}$ and a tax plan $\{\tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}\}$ that satisfies the govt budget constraint.

5. Competitive equilibrium with distorting taxes

Definition 1

A competitive equilibrium with distorting taxes is

- *a budget-feasible allocation*
- *a feasible allocation*
- *a price system*

such that, given the price system and the govt policy

- *the allocation solves the hh problem*
- *the allocation solves the firm problem*

- The hh intertemporal budget constraint (*ibc*) is

$$\sum_{t=0}^{\infty} q_t \left((1+\tau_{ct})c_t + (k_{t+1} - (1-\delta)k_t)\right) \leq \sum_{t=0}^{\infty} q_t \left(\eta_t k_t - \tau_{kt}(\eta_t - \delta)k_t + (1-\tau_{nt})w_t n_t - \tau_{ht}\right) \tag{2.4}$$

- Rewrite the terms in blue as follows (on the right-hand side of the *ibc*)
  - terms in k_0:

$$q_0(1-\delta) + q_0\eta_0 - q_0\tau_{k0}(\eta_0 - \delta) = ((1-\tau_{k0})(\eta_0 - \delta) + 1)q_0$$

  - terms in k_t:

$$-q_{t-1} + q_t(1-\delta) + q_t\eta_t - q_t\tau_{kt}(\eta_t - \delta) = \underbrace{((1-\tau_{kt})(\eta_t - \delta) + 1)q_t}_{\text{return on capital}} - \underbrace{q_{t-1}}_{\text{cost of capital}}$$

- Therefore, the budget constraint rewrites

$$\begin{aligned}\sum_{t=0}^{\infty} q_t(1+\tau_{ct})c_t \leq & \sum_{t=0}^{\infty} q_t(1-\tau_{nt})w_t n_t - \sum_{t=0}^{\infty} q_t \tau_{ht} \\ & + \sum_{t=0}^{\infty}(((1-\tau_{kt})(\eta_t-\delta)+1)q_t - q_{t-1})k_t \\ & + ((1-\tau_{k0})(\eta_0-\delta)+1)q_0 k_0 \\ & - \lim_{T\to\infty} q_T k_{T+1} \qquad (5.1)\end{aligned}$$

- Hh would be happy to have the highest possible right-hand side of (5.1)
- But this rhs must be bounded in equilibrium (because resources are finite) $\rightsquigarrow$ this is putting restictions on equilibrium prices.

5.1. The hh: no-arbitrage condition and asset-pricing formula

- Take the term in k_t: $\rho_t = \Big(\big((1-\tau_{kt})(\eta_t-\delta)+1\big)q_t - q_{t-1}\Big)$
  - × if $\rho_t > 0$: hh can:
    - buy in $t-1$ arbitrarily large k_t with present value $q_{t-1}k_t$
    - sell in t rental services and undepreciated part to obtain a present value income of $(((1-\tau_{kt})(\eta_t-\delta)+1)q_t)k_t$
    - as $\rho_t > 0$, thus gives an arbitrarily large benefit
    - the rhs of (5.1) would then be unbounded $\rightsquigarrow$ not an equilibrium.
  - × if $\rho_t < 0$: hh can does the reverse:
    - short-sell in $t-1$ at price q_{t-1}
    - deliver in t buy buying at price $((1-\tau_{kt})(\eta_t-\delta)+1)q_t$
    - again, the rhs of (5.1) would be unbounded $\rightsquigarrow$ not an equilibrium
- Therefore, by no-arbitrage

$$\frac{q_t}{q_{t+1}} = (1-\tau_{kt+1})(\eta_{t+1}-\delta)+1 \qquad \forall t \geq 0 \tag{5.2}$$

- and no possibility to short-shell at $+\infty$:

$$\lim_{T\to\infty} q_T k_{T+1} = 0$$

- Rewriting (5.2):

$$\underbrace{\eta_{t+1}}_{\text{user cost of capital}} = \underbrace{\delta}_{\text{depreciation}} + \underbrace{\left(\frac{1}{1-\tau_{kt+1}}\right)}_{\text{taxes}} \underbrace{\left(\frac{q_t}{q_{t+1}} - 1\right)}_{\text{capital gains or losses}} \tag{5.4}$$

5.3. Hh *foc*

$$\max \mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) + \mu \; ibc$$

- Hh are indifferent about the level of k_t as long as the no-arbitrage condition holds
- *foc* for c_t and n_t:

$$\beta^t U_{1t} = \mu q_t (1 + \tau_{ct}) \qquad (5.5a)$$
$$\beta^t U_{2t} = \mu w_t (1 - \tau_{nt}) \qquad (5.5b)$$

  assuming an interior solution $n_t < 1$.
- We see that only μq_t matters, not μ and q_t separately $\rightsquigarrow$ once can choose a numéraire, or can arbitrarily normalize $\mu = 1$

- Assume $U(c_t, 1 - n_t) = u(c_t) + v(1 - n_t)$
- *foc* wrt to c_t:

$$\mu q_t = \beta^t \frac{u'(c_t)}{1 + \tau_{ct}}$$

- $\{q_t\}$ and therefore the term structure can be computed if we observe $\{c_t\}$ $\rightsquigarrow$ CCAPM
- Govt policy $\{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}\}$ affects equilibrium $\{c_t\}$, and therefore the term structure.

- Firm value is

$$V_0 = \sum_{t=0}^{\infty} q_t (F(k_t, n_t) - w_t n_t - \eta_t k_t)$$

- Because of homegeneity of degree 1 (Euler theorem):

$$V_0 = \sum_{t=0}^{\infty} q_t ((F_{nt} - w_t) n_t + (F_{kt} - \eta_t) k_t)$$

- By no-arbitrage:

$$\begin{aligned} \eta_t &= F_{kt} \\ w_t &= F_{nt} \end{aligned} \tag{5.7}$$

6. Computing equilibria

- $\{g_t,\tau_t\} = \{g_t,\tau_{ct},\tau_{nt},\tau_{kt}\}$ is exogenous
- $\sum_{t=0}^{\infty} q_t \tau_{ht}$ is endogenous and makes sure that the govt intertemporally balances its budget.

6.1. Inelastic labor supply

- assume $U(c, 1-n) = u(c)$ and hh inelastically supply $n = 1$ (normalization)
- Define $f(k) = F(k, 1)$
- Feasibility writes

$$k_{t+1} = (1-\delta)k_t + f(k_t) - g_t - c_t \tag{6.1}$$

- Note that $F(k, n) = nF(k/n, 1) = nf(\widehat{k})$ with $k/n = \widehat{k})$
- One then has:

$$F_k = \frac{\partial[nF(k/n, 1)]}{\partial k} = n \times \frac{1}{n} \times \frac{\partial F(k/n, 1)}{\partial(k/n)} = f'(\widehat{k})$$

and

$$F_n = \frac{\partial[nF(k/n, 1)]}{\partial n} = n \times \frac{-k}{n^2} \times \frac{\partial F(k/n, 1)}{\partial(k/n)} + F(k/n, 1) = f(\widehat{k}) - \widehat{k}f'(\widehat{k})$$

- and when $n = 1$, $F_k = f'(k)$ and $F_n = f(k) - kf'(k)$

6. Computing equilibria
Some substitutions

- Take resource constraint

$$k_{t+1} = (1-\delta)k_t - g_t - c_t$$

- Obtain c_t and replace in the *foc*

$$\beta^t u'(c_t) = \mu q_t (1+\tau_{ct})$$

- Obtain q_t and q_{t+1} and replace in the no-arbitrage condition

$$\frac{q_t}{q_{t+1}} = (1-\tau_{kt+1})(\eta_{t+1} - \delta) + 1$$

where η_{t+1} is replaced using the no-arbitrage condition $\eta_t = F_{kt}$

- We then obtain a nonlinear second order difference equation in k_t

$$
\begin{aligned}
\frac{u'\left(f(k_t + (1-\delta)k_t - g_t - k_{t+1}\right)}{(1-\tau_{ct})} \quad &- \quad \beta \frac{u'\left(f(k_{t+1} + (1-\delta)k_{t+1} - g_{t+1} - k_{t+2}\right)}{(1-\tau_{ct+1})} \\
&\times \quad \left((1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right) \\
&= \quad 0
\end{aligned}
\tag{6.2}
$$

- initial condition k_0
- terminal condition $\lim_{T\to\infty} q_T k_{T+1} = 0$
- for given gvt policy
- (6.2) can be rewritten as

$$
u'\left(c_t\right)) = \beta u'(c_{t+1}) \frac{(1-\tau_{ct})}{(1-\tau_{ct+1}} \left((1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right) \tag{6.3}
$$

- Let $z_t = \{g_t, \tau_{kt}, \tau_{ct}\}$ be the sequence of exogenous variables
- (6.2) can be written as

$$H(k_t, k_{t+1}, k_{t+2}, z_t, z_{t+1}) = 0 \tag{6.4}$$

- For the steady state to be relevant, we look at cases where

$$\lim_{t\to\infty} z_t = \bar{z} \tag{6.5}$$

- At the steady state, we have

$$H(\bar{k}, \bar{k}, \bar{k}, \bar{z}, \bar{z}) = 0 \tag{6.6}$$

6.2. Equilibrium steady state

- (6.3) writes at the steady state

$$u'(\overline{c}) = \beta u'(\overline{c}) \frac{(1-\overline{\tau}_c)}{(1-\overline{\tau}_c)} \left((1-\overline{\tau}_k)(f'(\overline{k})-\delta)+1\right)$$

which gives

$$1 = \beta \left((1-\overline{\tau}_k)(f'(\overline{k})-\delta)+1\right) \tag{6.3b}$$

- Note: $\overline{\tau}_c$ does not distort $\overline{k}$
- With $\frac{1}{\beta} = 1+\rho$, steady state capital is pinned down by

$$f'(\overline{k}) = \delta + \frac{\rho}{1-\overline{\tau}_k}$$

6.3. Computing the equilibrium path with the shooting algorithm

- We want to solve the below difference equation system:

$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1}) \frac{(1-\tau_{ct})}{(1-\tau_{ct+1})} \left((1-\tau_{kt+1})(f'(k_{t+1})-\delta)+1\right) && \text{(Euler equation)} \\
k_{t+1} &= (1-\delta)k_t - g_t - c_t && (6.8a)
\end{aligned}
$$

  with boundary conditions

$$
\begin{cases}
k_0 \text{ given} \\
\lim_{T\to\infty} \beta^T \dfrac{u'(c_T)}{(1+\tau_{cT})} k_{T+1}
\end{cases}
$$

  where we have used $\beta^T u'(c_T) = \mu q_t (1+\tau_{ct})$
- Shooting algorithm:
  - × Take terminal period S large but finite
  - × Impose $k_S \approx \overline{k}$
  - × For given c_0, iterate the difference system forward starting from (k_0, c_0) and compute k_S
  - × Try many values of c_0
  - × Solution is found for the c_0 such that $k_S \approx \overline{k}$

- Once this is done, find $\{\tau_{ht}\}$ such that the govt budget constraint is satisfied
- Then compute prices using

$$
\begin{aligned}
q_t &= \beta^t \frac{u'(c_t)}{u'(c_{t+1})} && (6.8b) \\
\eta_t &= f'(k_t) && (6.8c) \\
w_t &= f(k_t) - k_t f'(k_t) && (6.8d) \\
\overline{R}_{t+1} &= \frac{1+\tau_{ct}}{1+\tau_{ct+1}} \left((1-\tau_{kt+1})((f'(k_{t+1})-\delta)+1\right) && (6.8e) \\
&= \frac{1+\tau_{ct}}{1+\tau_{ct+1}} R_{t,t+1} \\
R^{-1}_{t,t+1} &= m_{t,t+1} = \beta \frac{u'(c_t)}{u'(c_{t+1})} \frac{1+\tau_{ct}}{1+\tau_{ct+1}} && (6.8f) \\
r_{t,t+1} &= R_{t,t+1} - 1 = (1-\tau_{kt+1})(f'(k_{t+1})-\delta) && (6.8g) \\
u'(c_t) &= \beta u'(c_{t+1}) \overline{R}_{t+1} && (6.8h)
\end{aligned}
$$

- if $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, then (6.8h) becomes

$$\log \frac{c_{t+1}}{c_t} = \gamma^{-1} \log \beta + \gamma^{-1} \log \overline{R}_{t+1} \tag{6.9}$$

- (6.9): consumption growth varies with distorted real interest rate.

- What we have just done is to implement the shooting algorithm taking as given $\{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}\}$.
- Then, once prices and quantities are obtained, $\{\tau_{ht}\}$ is set such that $\sum_{t=0}^{\infty} q_t \tau_{ht}$ balances the govt budget constraint.
- We can do this two-step computation because $\{\tau_{ht}\}$ are nowhere in equations (6.8)
- The timing of $\{\tau_{ht}\}$ is irrelevant $\rightsquigarrow$ Ricardian equivalence

6.7. When no lump-sum taxes are available

- Then, an additional step is needed in the algorithm: making sure that the govt budget constraint is satisfied.
- Algorithm given a sequence of $\{g_t\}$:
  - × Assume sequence of taxes $\{\tau_{ct}, \tau_{nt}, \tau_{kt}\}$
  - × solve for the equilibrium using the shooting algorithm
  - × check if the budget constraint of the govt is satisfied
  - × If not, adjust taxes and repeat.

8. Effect of taxes on equilibrium allocations and prices

- τ_c, τ_n and τ_k are distortionary, meaning that hh can affect their tax payments by altering their decisions.
- τ_h is non distortionary.

- Suppose $\tau_c = 0$, $\tau_n = 0$ and $\tau_k = 0 \rightsquigarrow \tau_h$ does not enter anywhere in (6.8)
- The timing of $\{\tau_{ht}\}$ is irrelevant, only $\sum q_t \tau_{ht}$ matters in govt and hh intertemporal budget constraints.
- This is Ricardian equivalence

8.2. When labour supply is inelastic

- τ_n is not distorting
- Constant τ_c is not distorting
- Variations in τ_c are distorting
- Capital taxation τ_k is distorting

9. Transition experiments with inelastic labour supply

- Assume
  - $U(c, 1-n) = u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, $f(k) = k^{\alpha}$
  - $\alpha = 1/3$, $\delta = 0.2$, $\beta = .95$, $g = 0.2$
  - $\gamma = 2$ or $\gamma = 0.2$
- First we do a foreseen once-for-all increase in g, τ_c, τ_k
- The change is announced a $t = 0$ and takes place at $t = 10$, and the economy was at the steady state before 0.
- Although no change is implemented before $t = 10$, the economy reacts on impact
- Why? Because hh wants to smooth consumption $\rightsquigarrow$ they adjust their savings from period 0 and onwards $\rightsquigarrow$ prices and quantities move at time 0.
- Two forces are at play in the dynamics:
  - discounting of the future before T
  - transient dynamics after T

  and these two forces are interrelated (see later)

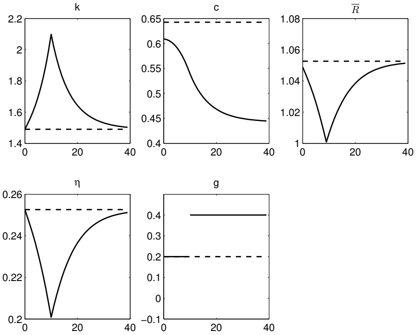


Figure 11.9.1: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$. The dashed line is the original steady state.

- The steady state level of k is unaffected (see (6.3b))
- $g \nearrow \rightsquigarrow c \searrow$
- from 0 to 10: $c \searrow \rightsquigarrow k \nearrow$ (because $g \rightarrow$)
- Initial negative wealth effect of c (because $\sum q_t \tau_{ht} \nearrow$)
- The dynamics of $\overline{R}$ makes the hh choosing a non flat c profile.
- Both feedforward and feedback dimension in the response of the economy (more on this later)

Foreseen permanent increase in g, $\gamma = 2$ or 0.2

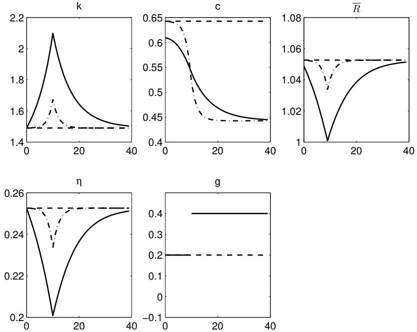


Figure 11.9.2: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$. The dashed lines show the original steady state. The solid lines are for $\gamma = 2$, while the dashed-dotted lines are for $\gamma = .2$

- More willingness to smooth consumption when $\gamma = 2$ as compared to when $\gamma = 0.2$
- When γ is small (the limit would be linear utility), c becomes the mirror image of g
- Less feedforward and less feedback effect $\rightsquigarrow$ the two dimensions are related (see later)

9. Transition experiments with inelastic labour supply

Foreseen permanent increase in g, asset prices

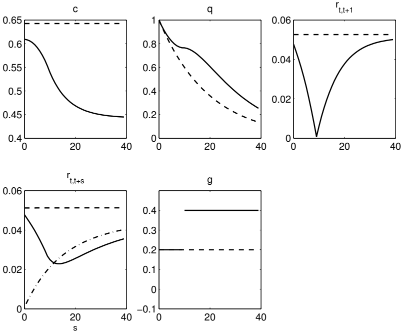


Figure 11.9.3: Response to foreseen once-and-for-all increase in g at $t = 10$. From left to right, top to bottom: $c, q, r_{t,t+1}$ and yield curves $r_{t,t+s}$ for $t = 0$ (solid line), $t = 10$ (dash-dotted line) and $t = 60$ (dashed line); term to maturity s is on the x axis for the yield curve, time t for the other panels.

- $q_t = \beta^t c_t^{-\gamma}$
- Short rate $r_{t,t+1} = -\log \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$
- q_t: price of future consumption is higher in the future (when g is higher)
- Term structure at 10 periods: upward sloping because the growth rate of c is expected to increase (to be less negative)
- Term structure at time 0 is U shaped.

9. Transition experiments with inelastic labour supply

Foreseen permanent increase in τ_c

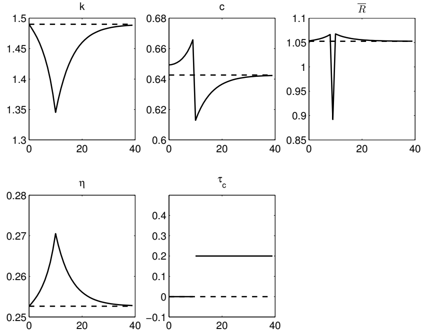


Figure 11.9.4: Response to foreseen once-and-for-all increase in τ_c at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \tau_c$.

- $$u'(c_t)) = \beta u'(c_{t+1}) \frac{(1 - \tau_{ct})}{(1 - \tau_{ct+1}} \left((1 - \tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right) \qquad (6.3)$$
- Anticipated decrease in $\frac{(1 - \tau_{ct})}{(1 - \tau_{ct+1}}$ $\equiv$ anticipated increase in τ_k, as seen in (6.3)
- The hh frontloads consumption, by $\searrow c$
- No effect on the steady state
- After T, no more anticipation effect $\rightsquigarrow$ transient dynamics when starting with low k

9. Transition experiments with inelastic labour supply

Foreseen permanent increase in τ_k

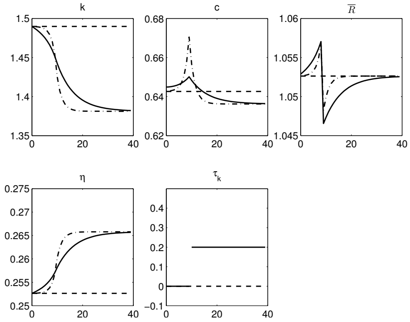


Figure 11.9.5: Response to foreseen increase in τ_k at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \tau_k$. The solid lines depict equilibrium outcomes when $\gamma = 2$, the dashed-dotted lines when $\gamma = .2$.

- Lower final steady state $\rightsquigarrow$ some capital can be eaten in the transition $\rightsquigarrow$ $c \nearrow$ before period 10.
- After 10, transient dynamics from a higher that steady state stock of capital.

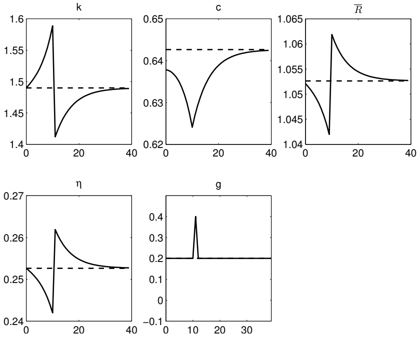


Figure 11.9.6: Response to foreseen one-time pulse increase in g at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, g$.

- Again, the anticipation effect is at play before 10
- Desire to smooth c
- in 10, govt takes out some good for g, but c stays smooth $\rightsquigarrow$ investment adjusts by $\searrow$.

10. Linear approximation

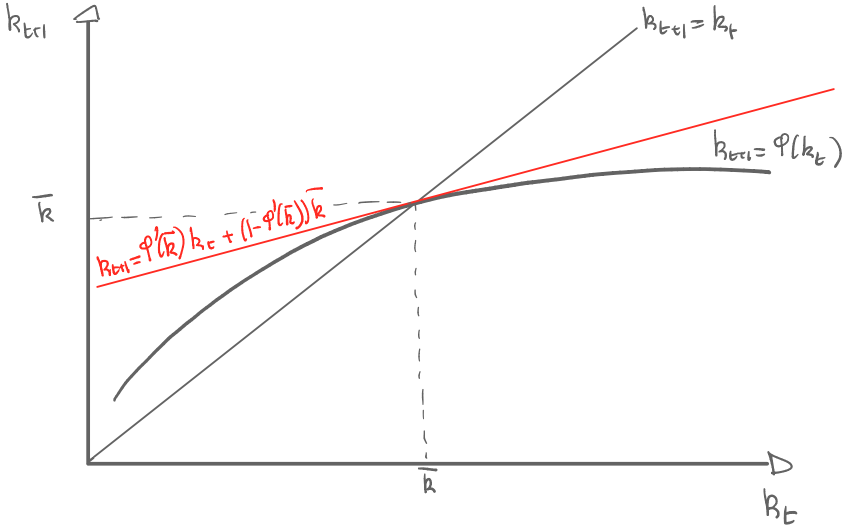
- Shooting algorithm can be tricky in larger models
- Useful to look at the solution of a linear approximation (one can also do log-linear)
- idea: Assume the model is

$$k_{t+1} = \varphi(k_t)$$

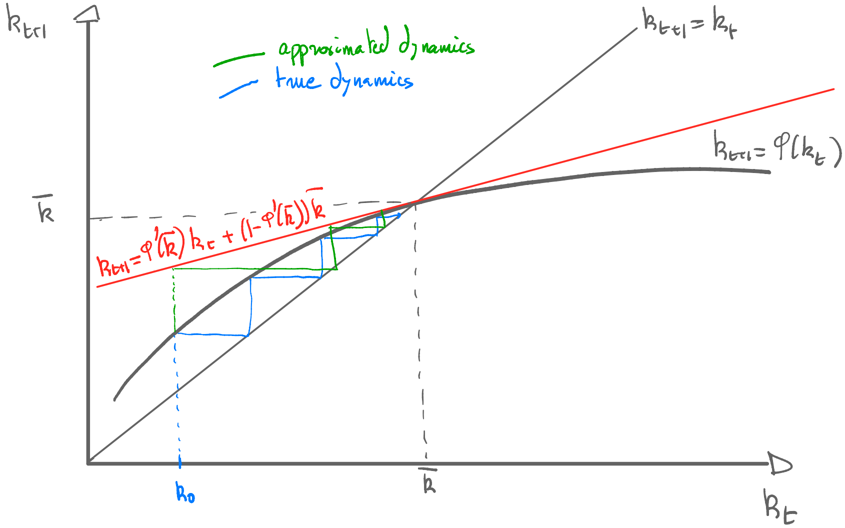
- The steady state is $\overline{k} = \varphi(\overline{k})$
- Linear approximation:

$$(k_{t+1} - \overline{k}) \approx \varphi'(\overline{k})(k_t - \overline{k})$$

10. Linear approximation



10. Linear approximation



- Let's show an important result: the model solution can be partitioned into a feedback and an feedforward part.
- Model is

$$H(k_t, k_{t+1}, k_{t+2}, z_t, z_{t+1}) = 0$$

- The steady state is given by

$$H(\overline{k}, \overline{k}, \overline{k}, \overline{z}, \overline{z}) = 0$$

- Linear approximation is

$$\begin{aligned} & H_{k_t} \times (k_t - \overline{k}) + H_{k_{t+1}} \times (k_{t+1} - \overline{k}) + H_{k_{t+2}} \times (k_{t+2} - \overline{k}) \\ & + H_{z_t} \times (z_t - \overline{z}) + H_{z_{t+1}} \times (z_{t+1} - \overline{z}) \\ & = 0 \end{aligned} \tag{10.1}$$

with $H_{k_t} = H_{k_t}(\overline{k}, \overline{k}, \overline{k}, \overline{z}, \overline{z}), ...$

- Rewrite (10.1) as

$$\phi_0 k_{t+2} + \phi_1 k_{t+1} + \phi_2 k_t = A_0 + A_1 z_t + A_2 z_{t+1} \tag{10.2}$$

  or

$$\phi(L) k_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1} \tag{10.3}$$

- We want to solve this equation, i.e. find k_{t+1} as a function of past endogenous variables (k_{t-j}) and exogenous variables z (past, present or future as there is here no uncertainty)
- To do so, we will manipulate and transform the characteristic polynomial $\phi(L) = \phi_0 + \phi_1 L + \phi_2 L^2$
- Let $\mu_{1,2}$ be the two roots of ϕ (the solutions to $\phi(L) = 0$). Assume they are non-zero, real and distinct (can be proved in some environments)
- We have $\phi(L) = \phi_2(\mu_1 - L)(\mu_2 - L)$ and $\mu_1 \mu_2 = \phi_0/\phi_2$.

- Write $\mu_i - L = \mu_i \left(1 - \frac{1}{\mu_i} L\right)$ so that $\phi(L)$ can be written

$$\underbrace{\phi_2 \mu_1 \mu_2}_{\phi_0} \left(1 - \frac{1}{\mu_1} L\right) \left(1 - \frac{1}{\mu_2} L\right)$$

  Denote $\lambda_i = \frac{1}{\mu_i}$ to obtain

$$\phi(L) = \phi_0 (1 - \lambda_1 L)(1 - \lambda_2 L)$$

- Let's assume (more on this later) that $|\lambda_1| > 1$, $|\lambda_2| < 1$

- Because $|\lambda_1| > 1$,

$$(1-\lambda_1 L)^{-1} = \sum_{j=0}^{\infty} \lambda_1^j L^j \qquad \text{diverges}$$

- We can flip this infinite sum:

$$(1-\lambda_1 L) = -\lambda_1 L \left(1 - \lambda_1^{-1} L^{-1}\right)$$

and

$$\left(1-\lambda_1^{-1} L^{-1}\right)^{-1} = \sum_{j=0}^{\infty} \lambda_1^{-j} L^{-j} \qquad \text{converges}$$

- Recall that $L^{-1} x_t = x_{t+1}$
- $\sum_{j=0}^{\infty} \lambda_1^{-j} L^{-j}$ is a forward looking term, which corresponds to a discounted sum of future values, with discounting at rate λ_1^{-1}

- We can then rewrite $\phi(L)$ as follows:

$$\begin{aligned} \phi(L) &= \phi_0(1-\lambda_1 L)(1-\lambda_2 L) \\ &= \phi_0(-\lambda_1 L)\left(1-\lambda_1^{-1}L^{-1}\right)(1-\lambda_2 L) \end{aligned}$$

- and using $\phi_2 = \lambda_1\lambda_2\phi_0$:

$$\phi(L) = \frac{-\phi_2}{\lambda_2} L\left(1-\lambda_1^{-1}L^{-1}\right)(1-\lambda_2 L)$$

- so that (10.3)

$$\phi(L)k_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1} \tag{10.3}$$

writes

$$\frac{-\phi_2}{\lambda_2}\left(1-\lambda_1^{-1}L^{-1}\right)(1-\lambda_2 L)Lk_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1} \tag{10.6}$$

$$\frac{-\phi_2}{\lambda_2}\left(1-\lambda_1^{-1}L^{-1}\right)(1-\lambda_2 L)Lk_{t+2} = A_0 + A_1 z_t + A_2 z_{t+1}$$

- Put the blue term on the right-hand side of the equation:

$$\underbrace{(1-\lambda_2 L)k_{t+1}}_{\substack{\text{Transient dynamics,}\\ \text{“feedback”,}\\ \text{“backward looking”}}} = \underbrace{\frac{-\lambda_2\phi_2^{-1}}{\left(1-\lambda_1^{-1}L^{-1}\right)}A_0 + A_1 z_t + A_2 z_{t+1}}_{\substack{\text{Expectational dynamics,}\\ \text{“feedforward”,}\\ \text{“forward looking”}}} \qquad (10.7)$$

- (10.7) can be more explicitly written as

$$k_{t+1} = \lambda_2 k_t - \lambda_2 \phi_2^{-1} \sum_{j=0}^{\infty} (\lambda_1)^{-j} \left[A_0 + A_1 z_{t+j} + A_2 z_{t+j+1} \right] \qquad (10.8)$$

- $(\lambda_1)^{-j}$ is the rate at which expectations about the future are discounted
- The derivation relies on the fact that $|\lambda_1| > 1$ and $|\lambda_2| < 1$.
- Whether this is true or not depends on the economic environment.
- It is true in the neoclassical growth model we are working with.

- k_0 is given
- In the linearized model, k_1 (or equivalently c_0) is chosen looking at the whole future.
- It corresponds in the shooting algorithm to the choice of the c_0 such that $k_S = \overline{k}$ after S periods, i.e. *in the future*

10.1. Relation between the λ_is

- When $\{g_t, \tau_t\} = 0 \; \forall t$, one can prove (a bit long) that

$$\lambda_1 \lambda_2 = 1/\beta$$

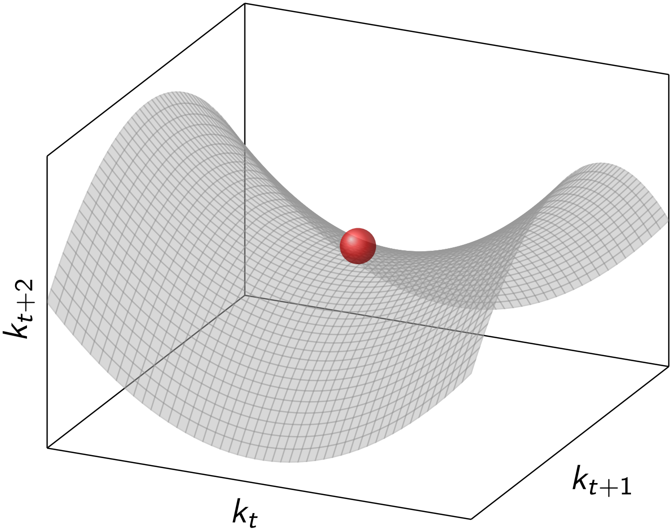
and that

$$|\lambda_1| > 1/\sqrt{\beta}$$

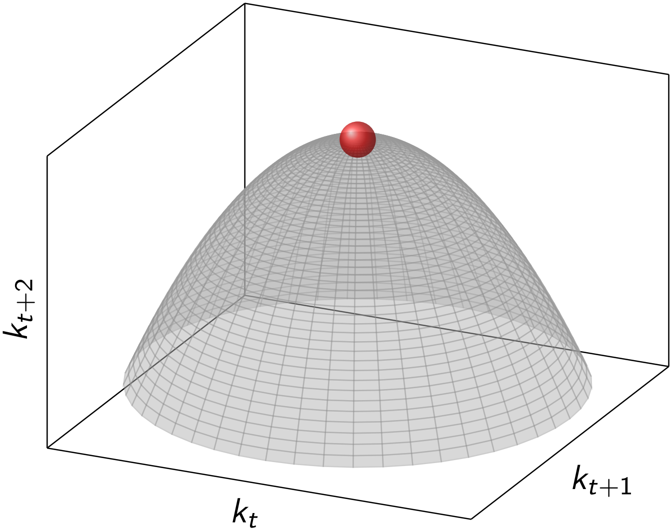
and

$$|\lambda_2| < 1/\sqrt{\beta}$$

- When $|\lambda_1| > 1$ and $|\lambda_2| < 1$, we have existence and uniqueness of the equilibrium dynamics
- This is a case in which for given k_0, there is a unique c_0 that satisfies non explosion.
- This is what we call *saddle-path stability*
- There are as many roots on the unit disc as predetermined variables $=$ Blanchard-Kahn [1980] condition

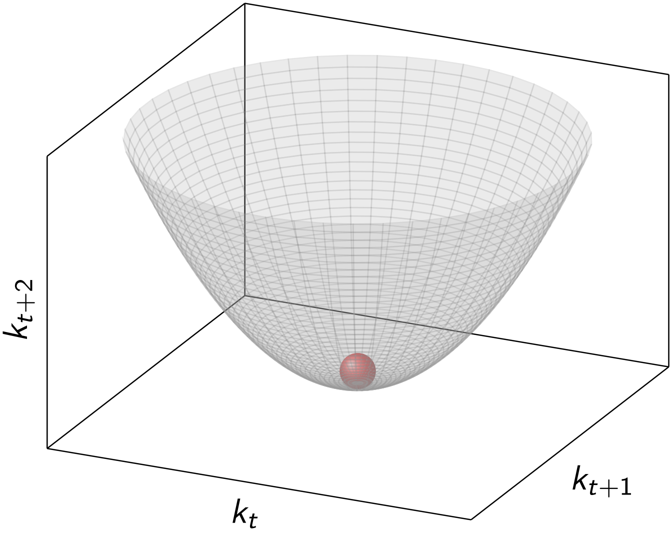

k_{t+2}
k_t
k_{t+1}

- When $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the model becomes explosive.
- One would need k_0 to jump to $\overline{k}$, but this is not possible as k_0 is predetermined.
- The economy will explode and at some point will violate resource constraint or positivity of c and k.
- The equilibrium does not exist.


k_{t+2}
k_t
k_{t+1}

- When $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the model is *indeterminate*: there is a continuum of paths that converge to the steady state.
- Given k_0, any c_0 id admissible.
- There are *sunspot equilibria*: if the economy believes that it should start from some $\widetilde{c}_0$, this is an equilibrium, and many $\widetilde{c}_0$ are admissible.

10.2. Existence and uniqueness of the equilibrium dynamics
Indeterminacy



- Given the above algebra, we can write the full approximate solution following a once-and-for-all jump in one forcing variable.
- Assume that the economy is initially at the steady state, that we normalize to $\overline{k} = \overline{z} = 0$
- Assume z is of dimension 1.
- The shock is :

$$z_t = \begin{cases} 0 & \text{if} \;\; t \leq T-1 \\ \widetilde{z} & \text{if} \;\; t \geq T-1 \end{cases}$$

- Define:

$$v_t = \sum_{i=0}^{\infty} \lambda_1^{-i} z_{t+i} = \begin{cases} \left(\frac{1}{\lambda_1}\right)^{T-t} \frac{1}{1-(\frac{1}{\lambda_1})} \widetilde{z} & \text{if } t \leq T-1 \\ \frac{1}{1-(\frac{1}{\lambda_1})} \widetilde{z} & \text{if } t \geq T-1 \end{cases} \tag{10.10}$$

$$h_t = \sum_{i=0}^{\infty} \lambda_1^{-i} z_{t+i+1} = \begin{cases} \left(\frac{1}{\lambda_1}\right)^{T-(t+1)} \frac{1}{1-(\frac{1}{\lambda_1})} \widetilde{z} & \text{if } t \leq T-1 \\ \frac{1}{1-(\frac{1}{\lambda_1})} \widetilde{z} & \text{if } t \geq T-1 \end{cases} \tag{10.11}$$

- Then using

$$k_{t+1} = \lambda_2 k_t - \lambda_2 \phi_2^{-1} \sum_{j=0}^{\infty} (\lambda_1)^{-j} \left[A_0 + A_1 z_{t+j} + A_2 z_{t+j+1} \right] \tag{10.8}$$

we obtain the solution

$$k_{t+1} = \begin{cases} \lambda_2 k_t - \dfrac{(\phi_0 \lambda_1)^{-1} A_0}{1 - \frac{1}{\lambda_1}} - \dfrac{(\phi_0 \lambda_1)^{-1} (\frac{1}{\lambda_1})^{T-t}}{1 - \frac{1}{\lambda_1}} (A_1 + A_2 \lambda_2) \widetilde{z} & \text{if } t \leq T - 1 \\ \lambda_2 k_t - \dfrac{(\phi_0 \lambda_1)^{-1}}{1 - \frac{1}{\lambda_1}} (A_0 + A_1 + A_2 \lambda_2) \widetilde{z} & \text{if } t \geq T - 1 \end{cases} \tag{10.10}$$

- Now $Y_t = F(K_t, A_t n_t)$
- $A_{t+1} = \mu A_t$
- Deflate quantity variables: $y_t = \frac{Y_t}{A_t n_t}$, $k_t = \frac{K_t}{A_t n_t}$, $c_t = \frac{C_t}{A_t n_t}$, $g_t = \frac{G_t}{A_t n_t}$
- $y_t = f(k_t) = F(k_t, 1)$

11. Growth

- Assume again that labour is inelastically supplied and $n_1 = 1$
- Feasibility:

$$k_{t+1} = \mu^{-1}(f(k_t) + (1-\delta)k_t - g_t - c_t) \tag{11.4}$$

- Euler:

$$u'(A_t c_t) = \beta u'(A_{t+1} c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \left((1-\tau_{kt+1})(f'(k_{t+1}) - \delta) + 1\right) \tag{11.5}$$

- With $u = \dfrac{c^{1-\gamma}}{1-\gamma}$,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta \mu^{-\gamma} \overline{R}_{t+1}$$

$\rightsquigarrow$ it is "as if" discount rate is now $\beta\mu^{-\gamma}$ $\rightsquigarrow$grwth increases discounting because marginal utility is decreasing (therefore future units of good are worth less with growth.

- At the steady state of the deflated economy (which corresponds to a *balanced growth path* of the non deflated economy):

$$f'(\overline{k}) = \delta + \left(\frac{(1+\rho)\mu^{\gamma} - 1}{1 - \tau_k} \right)$$

$\rightsquigarrow$ $\overline{k}$ is smaller when $\mu > 1$ (as compared to $\mu = 1$)

- We can solve the deflated economy using the shooting algorithm
- Then we can recover the levels by multiplying by A_t: $K_t = A_t k_t = A_0 \mu^t k(t)$, etc...
- Note that a permanent increase in μ

Foreseen permanent increase in μ

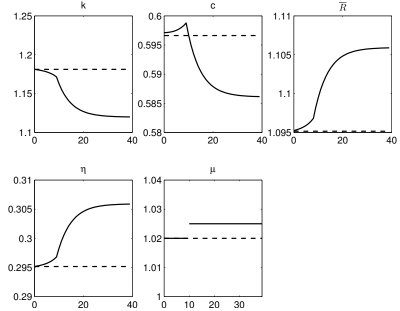


Figure 11.11.1: Response to foreseen once-and-for-all increase in rate of growth of productivity μ at $t = 10$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \mu$, where now k, c are measured in units of effective unit of labor.

- New steady state level of k is lower
- Consumption jumps immediately because people are wealthier.
- Increase in the gross return $\overline{R}$

11. Growth
Surprise permanent increase in μ

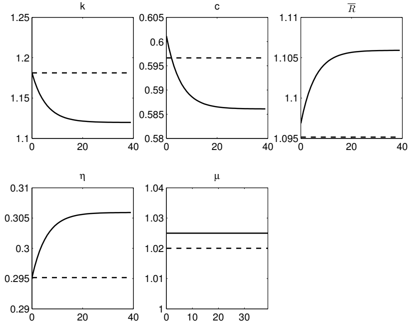


Figure 11.11.2: Response to increase in rate of growth of productivity μ at $t = 0$. From left to right, top to bottom: $k, c, \bar{R}, \eta, \mu$, where now k, c are measured in units of effective unit of labor.

- It looks very much like the transient part (after period 10) of the previous figure
- Increase in the gross return $\overline{R}$

12. Elastic Labour supply

$$\max \mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) + \mu \; ibc$$

- On top of the Euler equation, have an extra *foc*, which is the static consumption leisure decision.
- The two *foc* write

$$\begin{aligned}
U_1(F(k_t, n_t) + (1-\delta)k_t - g_t - k_{t+1}, 1 - n_t) &= \beta \left(\frac{1 + \tau_{ct}}{1 + \tau_{ct+1}} \right) \\
&\quad \times U_1(F(k_{t+1}, n_{t+1}) + (1-\delta)k_{t+1} - g_t - k_{t+2}, 1 - n_{t+1}) \\
&\quad \times [(1 - \tau_{kt+1})(F_k(k_{t+1}, n_{t+1}) - \delta) + 1] \\
\frac{U_2(F(k_t, n_t) + (1-\delta)k_t - g_t - k_{t+1}, 1 - n_t)}{U_1(F(k_t, n_t) + (1-\delta)k_t - g_t - k_{t+1}, 1 - n_t)} &= \left(\frac{1 - \tau_{nt}}{1 + \tau_{ct}} \right) F_n(k_t, n_t)
\end{aligned}$$

- We can again solve the model using the shooting algorithm or solving a linearized version.
- The steady state is now given by

$$\begin{aligned} \beta(1+(1-\tau_k)(F_k(\overline{k},\overline{n})-\delta)) &= 1 & (12.5) \\ \frac{U_2(\overline{c},1-\overline{n})}{U_1(\overline{c},1-\overline{n})} &= \left(\frac{1-\tau_n}{1+\tau_c}\right) F_n(\overline{k},\overline{n}) & (12.6) \\ \overline{c}+\overline{g}+\delta\overline{k} &= F(\overline{k},\overline{n}) & (12.7) \end{aligned}$$

- Given that $F_k(\overline{k},\overline{n}) = F_k(\frac{\overline{k}}{\overline{n}},1)$, (12.5) pins down $\widetilde{k} = \frac{\overline{k}}{\overline{n}}$
- (12.7) writes

$$\delta + \frac{\rho}{1-\tau_k} = f(\widetilde{k})$$

  $\rightsquigarrow$ only τ_k distorts $\widetilde{k}$.
- But τ_c and τ_n now distort the consumption/leisure decision.

- Assume $U(c, 1-n) = \log c + B(1-n)$ (Hansen-Rogerson preferences)
- B is chosen such that $0 < \overline{n} < 1$
- $\widetilde{k}$ can be computed from $f(\widetilde{k}) = \delta + \dfrac{\rho}{1-\tau_k}$
- The rest of the steady state can be computed as follows:
  - × (12.6) implies $\overline{c} = \dfrac{1}{B}\left(\dfrac{1-\tau_n}{1+\tau_c}\right)(f(\widetilde{k} - \widetilde{k}(f'(\widetilde{k}))$
  - × Then (12.7) implies $\overline{c} + \overline{g} + \delta\overline{k} = \overline{n}f(\widetilde{k})$ so that

$$\overline{n}(f(\widetilde{k}) - \delta\widetilde{k})^{-1}(\overline{c} + \overline{g}) \tag{12.14}$$

    which pins down $\overline{n}$
  - × Once $\overline{n}$ and $\widetilde{k}$ are known, $\overline{k} = \overline{n}\widetilde{k}$ can be obtained
- Let's assume same parameters values plus $B = 3$.

12. Elastic Labour supply
Unforeseen permanent increase in g

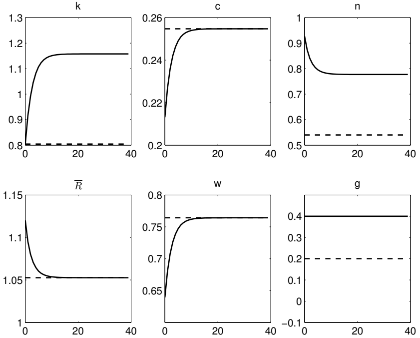


Figure 11.12.1: Elastic labor supply: response to unforeseen increase in g at $t = 0$. From left to right, top to bottom: $k, c, n, \bar{R}, w, g$. The dashed line is the original steady state.

- We have shown that $\overline{k}/\overline{n}$ and $\overline{c}$ not affected at the steady state
- (12.14) then implies that $\overline{n} \nearrow$ and therefore that $\overline{k} \nearrow$
- In the transition, $c \searrow$ and $n \nearrow$, which is bad for welfare.

12. Elastic Labour supply
Unforeseen permanent increase in τ_n

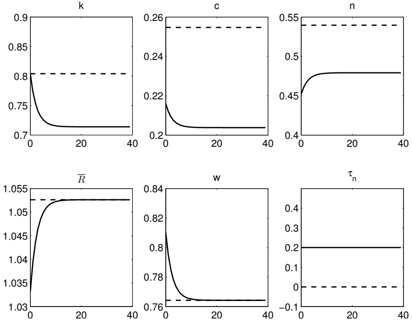


Figure 11.12.2: Elastic labor supply: response to unforeseen increase in τ_n at $t = 0$. From left to right, top to bottom: $k, c, n, \bar{R}, w, \tau_n$. The dashed line is the original steady state.

- Labour is discouraged $\rightsquigarrow$ the economy shrinks

12. Elastic Labour supply
Foreseen permanent increase in τ_n

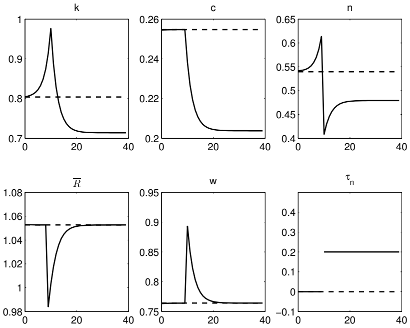


Figure 11.12.3: Elastic labor supply: response to foreseen increase in τ_n at $t = 10$. From left to right, top to bottom: $k, c, n, \bar{R}, w, \tau_n$. The dashed line is the original steady state.

- Long run effects are the same
- But in the short run $n, k \nearrow$ while c is flat
- It is worth working more (an saving) while labour is less taxed (before period 10)
- The impact of unexpected vs expected tax increase is in line with what is found in the data.
- Mertens and Ravn [2011], "Understanding the Effects of Anticipated and Unanticipated Tax Policy Shocks." Review of Economic Dynamics 14(1): 27-54. (Effect of tax <u>cuts</u>)

12. Elastic Labour supply

The Response to Tax Cuts in the US – Anticipated tax cuts are announced at date -6 and implemented at date 0 (Mertens and Ravn [2011]