2023-2024 – Econ 0107 – Macroeconomics I

Lecture 5 : Optimal Taxation with Commitment

(Chapter 16 in LJUNQVIST & SARGENT)

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- We study a dynamic optimal taxation problem with commitment = RAMSEY *problem*
- The solution to this problem is referred to as a RAMSEY *plan*
- The government's objective goal is to maximize households welfare subject to raising revenues through distortionary taxation.
- When designing an optimal policy, the government takes into account the competitive equilibrium reactions by consumers and firms to the tax system.

2. The economy

- No uncertainty
- Representative household (hh):

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) \tag{2.1}$$

- Technology

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1 - \delta)k_t \tag{2.3}$$

- F linearly homogenous of degree 1
- Govt:
  - spends exogenously $\{g_t\}$
  - finances with $\{\tau_{kt}\}$,$\{\tau_{nt}\}$ and one period real bonds $\{b_t\}$
  - Budget constraint

$$g_t = \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + \frac{b_{t+1}}{R_t} - b_t \quad , \quad \lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_{T+1} = 0 \tag{2.5}$$

  - Bonds are taxed exempted, no depreciation allowance

2.2. Households

- Choose $\{c_t, n_t, k_{t+1}, b_t\}$ to max

$$\begin{aligned}\max \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \Big(U(c_t, 1-n_t) \\ &+ \lambda_t \big((1-\tau_{nt}) w_t n_t + (1-\tau_{kt}) r_t k_t + (1-\delta) k_t + b_t - c_t - k_{t+1} - \frac{b_{t+1}}{R_t}\big)\Big)\end{aligned}$$

- *foc* :

$$\begin{aligned} U_{1t} &= \lambda_t && (2.7) \\ U_{2t} &= \lambda_t (1-\tau_{nt}) w_t && (2.8) \\ \lambda_t &= \beta \lambda_{t+1} ((1-\tau_{kt+1}) r_{t+1} + 1 - \delta) && (2.9) \\ \lambda_t &= \beta \lambda_{t+1} R_t && (2.10) \end{aligned}$$

- which implies

$$\begin{aligned} U_{2t} &= U_{1t} (1-\tau_{nt}) w_t && (2.11a) \\ U_{1t} &= \beta U_{1t+1} ((1-\tau_{kt+1}) r_{t+1} + 1 - \delta) && (2.11b) \\ R_t &= (1-\tau_{kt+1}) r_{t+1} + 1 - \delta && (2.12) \end{aligned}$$

(2.12) is a no-arbitrage condition

- We also have the transversality conditions

$$\lim_{T\to\infty}\left(\prod_{i=0}^{T-1} R_i^{-1}\right) k_{T+1} = 0 \tag{2.15}$$

$$\lim_{T\to\infty}\left(\prod_{i=0}^{T-1} R_i^{-1}\right) \frac{b_{T+1}}{R_T} = 0 \tag{2.16}$$

2.3. Representative firm

- Chooses $\{n_t, k_t\}$ to max $\{\Pi_t = F(k_t, n_t) - r_t k_t - w_t n_t\}$
- No-arbitrage:

$$\begin{aligned} r_t &= F_{kt} \\ w_t &= F_{nt} \end{aligned}$$

3. Ramsey problem

Definition 1

A feasible allocation is $\{k_t, c_t, n_t, g_t\}$ *that satisfies*

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1 - \delta)k_t \tag{2.3}$$

Definition 2

A price system is a nonnegative bounded sequence $\{w_t, r_t, R_t\}$

Definition 3

A government policy is a sequence $\{g_t, \tau_{kt}, \tau_{nt}, b_t\}$

3. Ramsey problem

Definition 4

A competitive equilibrium is a feasible allocation, a price system and a govt policy such that

1. *given price system and govt policy, allocation solves hh and firm problem*
2. *given price system and allocation, govt policy satisfies the sequence of budget constraints and transversality condition:*

$$g_t = \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + \frac{b_{t+1}}{R_t} - b_t; \quad \lim_{T \to \infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_{T+1} = 0 \tag{2.5}$$

3. Ramsey problem

Definition 5

Given k_0, b_0 and $\{g_t\}$, the Ramsey *problem is to choose $\tau_{kt}, \tau_{nt}\}$ so that the corresponding competitive equilibrium maximises utility.*

- We will impose the restriction that τ_{k0} is given and "small"
- Why? Because a trivial solution to the Ramsey problem is to use only τ_{k0} to finance $\{g_t\}$.
- Why? k_0 is given $\rightsquigarrow \tau_{k0}$ is not distortive.

4. Zero capital tax at the steady state

- Assume (without loss of generality) that the govt chooses $\widetilde{r}_t = (1 - \tau_{kt}) r_t$ and $\widetilde{w}_t = (1 - \tau_{nt}) n_t$ instead of τ_{kt} and τ_{nt}.
- Tax revenues are

$$\begin{aligned} \tau_{kt} r_t k_t + \tau_{nt} w_t n_t &= (r_t - \widetilde{r}_t) k_t + (w_t - \widetilde{w}_t) n_t \\ &= r_t k_t + w_t n_t - \widetilde{r}_t) k_t - \widetilde{w}_t n_t \\ &= \underbrace{F_{kt} k_t + F_{nt} n_t}_{F(k_t, n_t)} - \widetilde{r}_t k_t - \widetilde{w}_t n_t \end{aligned}$$

- Plug this expression in the govt budget constraint:

$$F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t = 0$$

- This equation collapses the govt *bc* and the firm's *focs*

4. Zero capital tax at the steady state

- The other constraints the govt is facing when choosing optimal taxes are :
  - Resource constraint:

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t$$

  - Hh *focs*:

$$\begin{aligned} U_{2t} &= U_{1t}(1-\tau_{nt})w_t \\ U_{1t} &= \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta) \end{aligned}$$

  - No-arbitrage condition

$$R_t = (1-\tau_{kt+1})r_{t+1} + 1 - \delta$$

- To derive the RAMSEY plan, the govt maximizes $\mathcal{L}$ by optimally choosing $\{k_{t+1}, c_t, n_t, \widetilde{w}_t, \widetilde{r}_t\}$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[U(c_t, n_t) \quad + \quad \psi_t\big(F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t\big)$$
$$+ \quad \theta_t\big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}\big) \tag{4.1}$$
$$+ \quad \mu_{1t}\big(U_{2t} - U_{1t}(1-\tau_{nt})w_t\big)$$
$$+ \quad \mu_{2t}\big(U_{1t} - \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta)\big)\Big]$$

4. Zero capital tax at the steady state

- The *foc* wrt k_{t+1} is

$$\underbrace{\theta_t}_{\substack{\text{cost of investing}\\ \text{one more unit}}} = \underbrace{\left(\beta\, \underbrace{\psi_{t+1}(F_{kt+1} - \widetilde{r}_{t+1})}_{\substack{\text{more tax revenues}\\ \text{tomorrow}}} + \underbrace{\theta_{t+1}(F_{kt+1} + 1 - \delta)}_{\substack{\text{more goods}\\ \text{tomorrow}}}\right)}_{\text{benefits of investing one more unit}} \tag{4.2}$$

- Assume that g_t stays constant after T so that $\exists$ a steady state to which the RAMSEY plan is converging.
- Using $r_t = F_{kt}$, (4.2) rewrites at the steady state:

$$\theta = \beta(\psi(r - \widetilde{r}) + \theta(r + 1 - \delta)) \tag{4.3}$$

4. Zero capital tax at the steady state

- Use the Hh Euler equation $U_{1t} = \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta)$ at the steady state:

$$1 = \beta(\widetilde{r} + 1 - \delta)$$

which implies

$$\begin{aligned} \beta(r + 1 - \delta) &= \beta(r - \widetilde{r} + \widetilde{r} + 1 - \delta) \\ &= \beta(r - \widetilde{r}) + \underbrace{\beta(\widetilde{r} + 1 - \delta)}_{1} \end{aligned}$$

- (4.3) becomes

$$\theta = \beta\psi(r - \widetilde{r}) + \theta \underbrace{\beta(r + 1 - \delta))}_{\beta(r-\widetilde{r})+1}$$

which implies

$$(\theta + \psi)(r - \widetilde{r}) = 0 \tag{4.5}$$

4. Zero capital tax at the steady state

$$(\theta + \psi)(r - \widetilde{r}) = 0 \tag{4.5}$$

- Since $\theta > 0$ and $\psi \leq 0$, one must have

$$r = \widetilde{r}$$

which implies

$$\tau_k = 0$$

Chamley-Judd result of no taxation of capital at the steady state.

Proposition 1

If $\exists$ a steady state Ramsey *allocation, the associated limiting tax rate on capital income is zero.*

$\approx$ Diamond-Mirlees no-taxation of intermediate goods result

6. Primal approach to the Ramsey problem

- Previously we have eliminated (τ_k, τ_n) and (w_t, r_t) and replace with $(\widetilde{w}_t, \widetilde{r}_t)$
- Let's go further and eliminate all taxes and prices.
  - × Dual approach: tax rates are the govt decisions
  - × Primal approach: the govt directly chooses quantities under the constraints imposed by competitive equilibrium

6. Primal approach to the RAMSEY problem

- The dual problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[U(c_t, n_t) + \psi_t \Big(F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t \Big)$$
$$+ \theta_t \big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} \big)$$
$$+ \mu_{1t} \big(U_{2t} - U_{1t}(1-\tau_{nt})w_t \big)$$
$$+ \mu_{2t} \big(U_{1t} - \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta) \big) \Big] \tag{4.1}$$

- Let's eliminate all prices and taxes
- Instead of the sequence of gvt budget constraints in blue, we will use the intertemporal budget constraint of the hh + the no-arbitrage condition (this is equivalent by WALRAS law):

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 (1-\tau_{nt}) w_t n_t + ((1-\tau_{k)}) r_0 + 1 - \delta) k_0 + b_0$$

Step 1 Obtain the *focs* of hh and firm + no-arbitrage pricing conditions and solve for $\{q_t^0, r_t, w_t, \tau_{kt}, \tau_{nt}\}$ as a function of $\{c_t, n_t, k_{t+1}\}$

Step 2 Substitute these expressions into the intertemporal *bc* if the hh $\rightsquigarrow$no more prices or taxes in that equation = *implementability condition*.

Step 3 Solve the Ramsey problem by maximizing welfare s.t. resource constraint and implementability condition.

Step 4 Once allocations are obtained, use the formulas of step 1 to obtain taxes.

6. Primal approach to the RAMSEY problem
Step 1

- Hh problem:

$$\max \sum \beta^t U(c_t, 1-n_t) + \lambda\left(-\sum_{t=0}^{\infty} q_t^0 + \sum_{t=0}^{\infty} q_t^0 (1-\tau_{nt}) w_t n_t + ((1-\tau_{k)}) r_0 + 1 - \delta) k_0 + b_0\right)$$

- *focs* are

$$\begin{aligned} \beta^t U_{1t} &= \lambda q_t^0 \\ \beta^t U_{2t} &= \lambda q_t^0 (1-\tau_{nt}) w_t \end{aligned}$$

- with $q_0^0 = 1$ (numèraire), we have

$$\begin{aligned} q_t^0 &= \beta^t \frac{U_{1t}}{U_{1,0}} \quad (6.3a) \\ (1-\tau_{nt}) w_t &= \frac{U_{2t}}{U_{1t}} \quad (6.3b) \end{aligned}$$

- Note that now taxes and prices are functions of quantities.

- Plus no-arbitrage conditions

$$
\begin{aligned}
\frac{q_t^0}{q_{t+1}^0} &= (1-\tau_{kt+1})r_{t+1} + 1 - \delta \\
r_t &= F_{kt} \\
w_t &= F_{nt}
\end{aligned}
$$

- Put (6.3a,b) and $r_0 = F_{k0}$) in the *ibc* of the hh

$$\sum_{t=0}^{\infty} \underbrace{q_t^0}_{\beta^t \frac{U_{1t}}{U_{1,0}}} c_t = \sum_{t=0}^{\infty} \underbrace{q_t^0}_{\beta^t \frac{U_{1t}}{U_{1,0}}} \underbrace{(1-\tau_{nt})w_t}_{\frac{U_{2t}}{U_{1t}}} n_t + \Big((1-\tau_{k0)}) \underbrace{r_0}_{F_{k0}} +1-\delta\Big) k_0 + b_0$$

which gives

$$\sum_{t=0}^{\infty} \beta^t (U_{1t} c_t - u_2 n_t) - A_0 = 0 \tag{6.5}$$

with

$$A_0 = U_{1,0}\Big(((1-\tau_{k0})F_{k0}k_0 + 1 - \delta)k_0 + b_0\Big) \tag{6.6}$$

- Let Φ be the Lagrange multiplier to the *ibc* (6.5)
- Define

$$V(c_t, n_t, \Phi) = U(c_t, 1 - n_t) + \Phi(U_{1t} c_t - U_{2t} n_t)$$

- Write the Lagrangian

$$\mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1 - \delta) k_t - c_t - g_t - k_{t+1} - \Phi A_0 \big) \Big)$$

- The govt problem is then to max $\mathcal{J}$ wrt $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ for given k_0, b_0, τ_{k0} $\rightsquigarrow$ only a choice of quantities in this problem.

- Interpretation: In that primal problem, the govt is choosing quantities under the constraint that these quantities can be decentralized with an appropriate set of optimal taxed (this is the *implementability condition*)

$$\max \mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} - \Phi A_0\big)\Big)$$

focs are:

$$\begin{array}{llcll}
c_t : & V_{ct} & = & \theta_t & \forall t \geq 1 \\
n_t : & V_{nt} & = & -\theta_t F_{nt} & \forall t \geq 1 \\
k_{t+1} : & \theta_t & = & \beta\theta_{t+1}(F_{kt+1} + 1 - \delta) & \forall t \geq 0 \\
c_0 : & V_{c0} & = & \theta_0 + \Phi \dfrac{\partial A_0}{\partial c_0} & \\
n_0 : & V_{n0} & = & -\theta_0 F_{n0} + \Phi \dfrac{\partial A_0}{\partial n_0} &
\end{array}$$

- One can eliminate θ_t to obtain:

$$\begin{aligned}
V_{ct} &= \beta V_{ct+1}(F_{kt+1} + 1 - \delta) && \forall t \geq 1 \quad (6.9a)\\
V_{nt} &= -V_{ct} F_{nt} && \forall t \geq 1 \quad (6.9b)\\
V_{c0} - \Phi \frac{\partial A_0}{\partial c_0} &= \beta V_{c1}(F_{k1} + 1 - \delta) && (6.9c)\\
V_{n0} &= \left(\Phi \frac{\partial A_0}{\partial c_0} - V_{c0} \right) F_{n0} + \Phi \frac{\partial A_0}{\partial n_0} && (6.9d)
\end{aligned}$$

plus the constraints

$$F(k_t, n_t) + (1-\delta)k_t = c_t + g_t + k_{t+1} \tag{6.10a}$$

$$\sum_{t=0}^{\infty} \beta^t (U_{1t} c_t - u_2 n_t) = A_0 \tag{6.10b}$$

$\rightsquigarrow$ solve for $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ and Φ

- Once allocations are obtained, we can find prices and taxes:

$$
\begin{aligned}
q_{0t} &= \beta^t \frac{U_{1t}}{U_{1,0}} && \rightsquigarrow q_{0t} \\
r_t &= F_{kt} && \rightsquigarrow r_t \\
w_t &= F_{nt} && \rightsquigarrow w_t \\
(1-\tau_{nt})w_t &= \frac{U_{2t}}{U_{1t}} && \rightsquigarrow \tau_{nt} \\
\frac{q_t^0}{q_{t+1}^0} &= (1-\tau_{kt+1})r_{t+1} + 1 - \delta && \rightsquigarrow \tau_{kt+1}
\end{aligned}
$$

- Assume $g_t = g \; \forall t \geq T$ and that there exists a steady state
- $V_{ct} \rightarrow \overline{V}_c$
- Then

$$V_{ct} = \beta V_{ct+1}(F_{kt+1} + 1 - \delta) \tag{6.9a}$$

implies $\beta(F_k + 1 - \delta)) = 1$

- We also have $q_{0t} = \beta^t \frac{U_{1t}}{U_{1,0}} \rightsquigarrow \lim_{t\to\infty} q_{0t}/q^0_{t+1} = 1/\beta$
- Then $\frac{q^0_t}{q^0_{t+1}} = (1 - \tau_{kt+1})r_{t+1} + 1 - \delta \rightsquigarrow \beta((1-\tau_k)F_k + 1 - \delta) = 1$
- This implies $\tau_k = 0$

7. Taxation of initial capital

- Up to now, we have assumed τ_{k0} given and small
- Assume now that τ_{k0} is a choice variable for the govt

$$\max \mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} - \Phi A_0 \big) \Big)$$

with

$$A_0 = U_{1,0} \Big(((1-\tau_{k0})F_{k0}k_0 + 1 - \delta)k_0 + b_0 \Big) \tag{6.6}$$

- We have

$$\frac{\partial \max \mathcal{J}}{\partial \tau_{k0}} = \Phi U_{1,0} F_{k0} k_0$$

7. Taxation of initial capital

$$\frac{\partial \max \mathcal{J}}{\partial \tau_{k0}} = \Phi U_{1,0} F_{k0} k_0$$

- Φ = utility cost of raising resources throught distortionary taxes
- As long as $\Phi > 0$, $\frac{\partial \max \mathcal{J}}{\partial \tau_{k0}} > 0$ and it is optimal to raise τ_{k0}
- $\tau_{k0} \nearrow \rightsquigarrow$ less distortive taxation $\rightsquigarrow \Phi \searrow$
- The govt will optimally increase τ_{k0} to the point where $\Phi = 0$ = First best
- In the first bes:
  - × $\tau_{kt} = \tau_{nt} = 0$
  - × The govt taxes $\tau_{k0} k_0$, lends it the the hh and pays g with the interest