UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

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This paper is suitable for candidates who attended classes for this module in the following academic year(s):

Year
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Additional material	
Special instructions	
Exam paper word count	

TURN OVER

University College London, 2022-2023
Econ 0107 – Macroeconomics I – Łukasz Rachel & Franck Portier

Midterm Assessment
Solution

Answer the four problems. Each problem carries 25% of the total mark.
Students are expected to spend a maximum of 5 hours on the paper.

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I – Human capital

This question analyzes an economy in which individuals accumulate human capital subject to idiosyncratic risk. Individuals accumulate human capital in a fashion similar to the Lucas (1988) endogenous growth model. There are two main differences: (a) this human capital accumulation is subject to idiosyncratic risk (which matters because individuals are risk averse), and (b) there is a continuum of individuals subject to this idiosyncratic risk rather than a representative agent.

Individuals have preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

where c_t is consumption and the period utility function $u(\cdot)$ is strictly increasing and strictly concave. Individuals are endowed with one unit of time. Human capital h_t accumulates according to

$$h_{t+1} = h_t(1 + g(s_t)),$$

where $s_t \in [0, 1]$ is the fraction of time devoted to human capital accumulation and $g(\cdot)$ is a strictly increasing function satisfying $g(0) = 0$. Given a time allocation s_t consumption is given by

$$c_t = z_t h_t (1 - s_t),$$

where z_t follows a Markov process with transition probabilities $F(z_{t+1}|z_t)$. The idea is that individual productivity fluctuates randomly over time even conditional on a given stock of human capital h_t (some days you're more productive than others; hopefully today is a high-z day for you!).

1 – Write the Bellman equation for the individual optimization problem.

The Bellman equation is

$$v(h, z) = \max_{s \in [0,1]} \; u(zh(1 - s)) + \beta \mathbb{E}[v(h', z')|z] \quad \text{s.t.} \quad h' = h(1 + g(s))$$

where the expectation operator $\mathbb{E}[\cdot]$ is more precisely given by

$$\mathbb{E}[v(h', z')|z] = \int v(h', z') dF(z'|z).$$

2 − Describe an algorithm for solving the Bellman equation you wrote in question (a). Present the algorithm as numbered steps that one could implement on a computer. As part of this description, please carefully discuss how to convert the problem with continuous state variable(s) into a problem that can be handled on a computer, e.g. via discretization. The algorithm should also feature a stopping criterion.

Value function iteration. The general steps are as follows. Guess an initial value function $v^0(h, z)$ and then for $\ell = 0, 1, 2...$ proceed as follows

(a) Given $v^\ell(h, z)$ and the process for productivity $F(z'|z)$, compute the expected continuation value $\mathbb{E}[v^\ell(h', z')|z] = \int v^\ell(h', z')dF(z'|z)$.

(b) Given the expected continuation value, compute a guess for the policy function

$$s^\ell(h, z) = \arg\max_{s\in[0,1]} \; u(zh(1 - s)) + \beta\mathbb{E}[v^\ell(h', z')|z] \quad \text{s.t.} \quad h' = h(1 + g(s))$$

(c) Given the guess for the policy function $s^\ell(h, z)$, update the value function

$$v^{\ell+1}(h, z) = u(zh(1 - s^\ell(h, z))) + \beta\mathbb{E}[v^\ell(h(1 + g(s^\ell(h, z))), z')|z]$$

(d) When $v^{\ell+1}(h, z)$ is "sufficiently close" to $v^\ell(h, z)$, then stop. For example, the criterion could be $||v^{\ell+1}(h, z) - v^\ell(h, z)|| < 10^{-8}$ where $|| \cdot ||$ is the sup norm.

On a computer you will typically want to work with a discrete state space, both for h and z. Therefore we typically discretize the state space for human capital as $h = \{h_1, ..., h_I\}$ and that for productivity as $z = \{z_1, ..., z_J\}$. The productivity process $F(z'|z)$ then becomes a finite-state Markov chain with some transition matrix $P = [p_{jj'}]$. Denoting $v_{i,j} = v(h_i, z_j)$, the value function becomes a vector of length $I \times J$. The discretized Bellman equation is

$$v(h_i, z_j) = \max_{s\in[0,1]} \; u(z_j h_i(1 - s)) + \beta \sum_{j'=1}^{J} v(h'_i, z_{j'})p_{jj'} \quad \text{s.t.} \quad h'_i = h_i(1 + g(s))$$

With this discretization, we can then follow the analogues of steps 1-4 above, e.g. step 1 becomes: Given $v^\ell_{i,j}$ and the process for productivity $P = [p_{jj'}]$, compute the expected continuation value $\sum_{j'=1}^{J} v(h'_i, z_{j'})p_{jj'}$.

3 − Derive the first-order equation for the time investment decision s. How do you expect the investment policy function for s to behave as a function of idiosyncratic productivity z? For this question, you can assume $u(c) = c^{1-\sigma}/(1 - \sigma), g(s) = \bar{g}s$. No need to solve the problem. It's sufficient to argue verbally using intuition or examples.

Ignoring the constraints $0 \le s \le 1$, the first-order condition is

$$u'(zh(1 - s))zh = \beta\mathbb{E}[v_h(h(1 + g(s)), z')|z]hg'(s)$$

The constraints $0 \le s \le 1$ additionally lead to the usual complementary slackness conditions. Next we speculate how the optimal policy function $s(h, z)$ behaves as a function of the state variable z. A useful special case is when $u(c) = c^{1-\sigma}/(1 - \sigma), g(s) = \bar{g}s$ and z is iid over time so that the optimal time investment satisfies

$$\frac{1}{1 - s}(zh)^{1-\sigma} = \beta\mathbb{E}[v_h(h(1 + g(s)), z')]h\bar{g}$$

This special case helps illustrate: how $s(h, z)$ varies with z is ambiguous and depends on

(a) income vs substitution effects as captured by the parameter σ. The substitution effect is that a high z means a high opportunity cost of allocating time toward human capital investment and therefore less investment. On the other hand, a high z means that the individual is richer and can therefore allocate more resources toward both consumption and investment. As σ increases and therefore the intertemporal elasticity of substitution $1/\sigma$ decreases, the strength of the income effect increases and that of the substitution effect decreases. When $\sigma = 1$ (and in the special case of an iid process for productivity z) the income and substitution effects offset and s is independent of z.

(b) the process for z. For example, when $\sigma = 1$ and z is iid over time s is independent of z. When $\sigma = 1$ and z is persistent so that $\mathbb{E}[v_h(h(1+g(s)), z')|z]$ is increasing in z, s is likely increasing in z. This is because a high z today predicts a high z tomorrow and therefore a high marginal value of human capital $v_h(h(1+g(s)), z')$.

4 – What can you say about the existence of a stationary distribution of human capital?

This model does not have a stationary distribution of human capital. This is because some individuals will choose $s > 0$ and the law of motion of human capital satisfies $h' = h(1+g(s)) \geq h$ with strict inequality when $s > 0$. Essentially this is a growth model, just like the Lucas endogenous growth model. The fact that individuals face idiosyncratic risk does not change this property.

5 – Solve this problem on a computer, assuming that $z = \{0.5, 1, 1.5\}$ with each outcome occurring with probability 1/3 every period, $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma = 2$, $g(s) = s$. *Hint 1: you may want to first re-express the problem so that* h_{t+1} *or* $\frac{h_{t+1}}{h_t} \in [1, 2]$ *is the control variable.* Plot the value function and the policy function. Verify your answers to part (3) by also presenting the solution for $\sigma = 1$. *Hint 2: You should submit the code and the results, even if they are unfinished or contain a bug. You are of course welcome to start from an existing piece of code and adjust it appropriately.*

II – An earthquake in a neoclassical growth model

Consider the optimal growth problem in continuous time

$$\begin{aligned} &\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt \\ \text{s.t.} \quad &\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - \delta k\left(t\right) - c\left(t\right) \text{ for each } t, \\ &c\left(t\right), k\left(t\right) \geq 0 \text{ and given } k\left(0\right), \end{aligned}$$

and $u(c)$ and $f(k)$ strictly increasing, strictly concave, and satisfying the Inada conditions.

1 – Write down the current-value Hamiltonian. What is the interpretation of the costate variable in this setting? (use the notation μ for the costate variable)

The CV Hamiltonian is

$$\hat{H}\left(t, c, k, \mu\right) = u\left(c\right) + \mu\left(f\left(k\right) - \delta k - c\right).$$

The costate variable μ is the marginal value of the state variable at time t.

2 – Write down the optimality conditions.

$\hat{H}_c = 0$, which implies

$$u'(c) = \mu.$$

$\hat{H}_k = \rho\mu - \dot{\mu}$, which implies

$$\rho\mu = \mu[f'(k) - \delta] + \dot{\mu}.$$

$\hat{H}_\mu = \dot{k}$, which implies

$$\dot{k} = f(k) - \delta k - c,$$

$\lim_{t\to\infty} e^{-\rho t}\mu\left(t\right)k\left(t\right) = 0$ for the candidate path, and (ii) $\lim_{t\to\infty} e^{-\rho t}\mu\left(t\right)\tilde{k}\left(t\right) \geq 0$ for any other feasible path $\left[\tilde{k}\left(t\right)\right]_{t=0}^{\infty}$.

3 – Derive the Euler equation.

Plugging condition 1 into 2, we obtain condition 1+2,

$$\begin{aligned}\rho u'(c) &= u'(c)[f'(k)-\delta] + \underbrace{\frac{du'(c)}{dt}}_{\dot{\mu}} \\ &= u'(c)[f'(k)-\delta] + u''(c)\underbrace{\frac{dc}{dt}}_{\dot{c}}.\end{aligned}$$

Rearranging terms, we obtain an Euler equation,

$$\frac{\dot{c}}{c} = \mathcal{E}(c)\left[f'(k)-\delta-\rho\right] \text{ where } \mathcal{E}(c) = \frac{-u'(c)}{cu''(c)}.$$

4 – In lectures on several occasions we have discussed a partial equilibrium consumption saving problem. The Euler equation in that case was $\frac{\dot{c}}{c} = \mathcal{E}(c)(r-\rho)$. We derived the conditions for consumption path to be either flat, increasing or decreasing forever. Does similar analysis apply to the model in this question? Why / why not?

The analysis applies but this is a GE model where the return to capital falls as capital is accumulated. Consumption thus settles at a unique steady state level, as is clear form the phase diagrams below.

5 – For this subquestion only, consider the decentralised equilibrium of this economy. What is the long-run elasticity of capital supply with respect to the interest rate in this economy? How does this elasticity compare to the one usually found in Bewley-Hugget-Aiyagari type models?

∞, which is greater than in BHA models.

6 – Assume $u(c) = \frac{1-e^{-ac}}{a}$ with $a > 0$. Draw a phase diagram in the k, c space. Illustrate the dynamics and draw the saddle path.

Standard.

7 – Calculate the steady state value of capital and consumption, assuming that the production function is $f(k) = k^\alpha$. How do the steady state values depend on a? Provide some intuition.

$k^* = \left[\frac{\alpha}{\delta+\rho}\right]^{\frac{1}{1-\alpha}}$ and $c^* = (k^*)^\alpha - \delta k^*$. These are independent of a; in steady state there is no risk and interest rate is constant so risk aversion and intertemporal elasticity do not affect the steady state but only the path along which the steady state is reached.

8 – Suppose the economy is in steady state. Then an earthquake wipes out around a third of the capital stock in the economy. Mark the position of the economy right after the earthquake on the phase diagram you have drawn. Explain briefly how the post-earthquake consumption is determined.

Consumption is on the (unchanged) saddle path. Otherwise system dynamics lead to outcomes that are infeasible or non-optimal.

9 – Suppose the economy is in steady state. Then there is a sudden increase in a. Interpret this shock (in one sentence). Trace out the dynamic path following the shock on the phase diagram. In a separate figure, plot the time paths of k and c. Explain which variables jump (if any) and why.

The saddle path becomes less steep. No variables react to this shock.

10 – Now assume that, starting from the steady state, the increase in a came together with an earthquake that wiped out around a third of the capital stock in the economy. Trace out the dynamic path following the shock on the phase diagram. On a separate figure, plot the time paths of k and c. Explain which variables jump (if any) and why.

The saddle path becomes less steep. Both capital and consumption drop on impact and then recover along the less steep saddle path.

11 – Assume that the increase in a came together with an earthquake that wiped out around a third of the capital stock in the economy, but the increase is temporary: a will remain high for T years and will then revert back to its pre-earthquake level. Trace out the dynamic path following the shock on the phase diagram. On a separate figure, plot the time paths of k and c. Explain the logic for the paths you have drawn and give some economic intuition.

The economy starts below the high-a saddle path, in such a way that at T the trajectory of the economy meets the low-a saddle path. Both c and k increase over time. Intuitively, it is optimal to save more today as the consumption smoothing motive is weaker, anticipating an increase in the elasticity of intertemporal substitution.

12 – Finally, draw a phase diagram in the k, μ space. Trace out the effect of a shock to a only, as in question **7** – above. Interpret your results.

In this space the saddle path is downward sloping. The shock lowers the arched curve. μ jumps down immediately, and k is unchanged at its steady state level. The intuition is that the marginal utility of consumption has fallen, and so the marginal value of capital is lower.

III – The yield curve

Consider a deterministic infinite horizon endowment economy with time-0 trading. There are N agents indexed by $i = 1, \ldots, N$ and endowments are $y^i = \{y_t^i\}_{t=0}^{\infty}$. Preferences are

$$U^i(c^i) = \sum_{t=0}^{\infty} \beta^t \log c_t^i.$$

We assume that agents trade assets in period 0. Asset "t" has a price q_t^0 and pays one unit of good in period t.

Let's first assume that $y_t^i = y \ \forall i$ and $\forall t$.

1 – Write the maximisation problem of agent i, denoting μ^i the Lagrange multiplier of the intertemporal budget constraint. Derive the first order condition with respect to c_t^i.

$$\max_{c^i} \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

$$\text{s.t} \quad \sum_{t=0}^{\infty} q_t^0 c_t^i \leq \sum_{t=0}^{\infty} q_t^0 y \qquad (\mu^i)$$

FOC is

$$\beta^t \frac{1}{c_t^i} = \mu^i q_t^0$$

2 – Define a competitive equilibrium for this economy. Solve for prices and quantities (you can use the normalisation $q_0^0 = 1$).

A competitive equilibrium is a sequence of quantities $\{c_t^i\}$ for all i and t and prices $\{q_t^0\}$ such that (i) given prices, consumption is optimal and (ii) markets clear –i.e.

$$\sum_{i=1}^{N} c_t^i = Ny \quad \text{for all } t.$$

As all agents are identical, the competitive equilibrium will be symetrical, and the resource constraint implies $c_t^i = y$ for all t and i. Using the household FOC in period 0 and t:

$$\frac{q_t^0}{q_0^0} = \frac{\beta^t}{\mu^i c_t^i} \frac{\mu^i c_0^i}{\beta^0} = \beta^t$$

Using the normalisation $q_0^0 = 1$, we obtain

$$q_t^0 = \beta^t.$$

3 – Define $r_{t,t+1}$ by

$$\frac{q_{t+1}^0}{q_t^0} = \frac{1}{1 + r_{t,t+1}} \approx e^{-r_{t,t+1}}$$

and $r_{0,t}$ by

$$q_t = e^{-r_{0,1}} e^{-r_{1,2}} \cdots e^{-r_{t-1,t}} = e^{-t r_{0,t}}$$

with $r_{0,t} = \frac{1}{t}(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t})$. Compute $r_{0,t}$ $\forall t$ and plot the yield curve –i.e $r_{0,t}$ as a function of t.

We have for all t

$$\frac{1}{1 + r_{t,t+1}} = \beta.$$

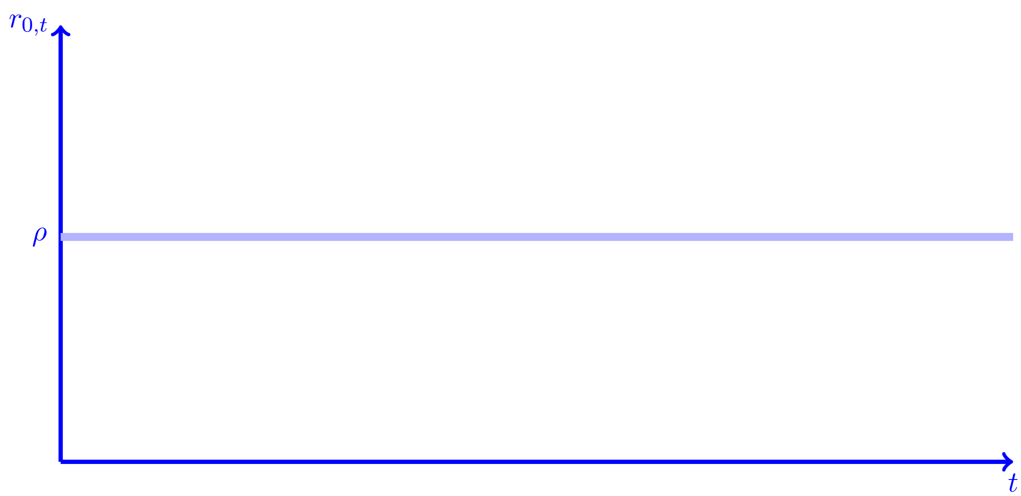
so that

$$r_{t,t+1} = \frac{1 - \beta}{\beta} \equiv \rho.$$

We therefore have

$$r_{0,t} = \rho$$

and the yield curve is flat.



4 – Assume now that $y_t^i = \gamma^t y$ with $\gamma > 1$. Compute $r_{0,t}$ $\forall t$ and plot the yield curve.

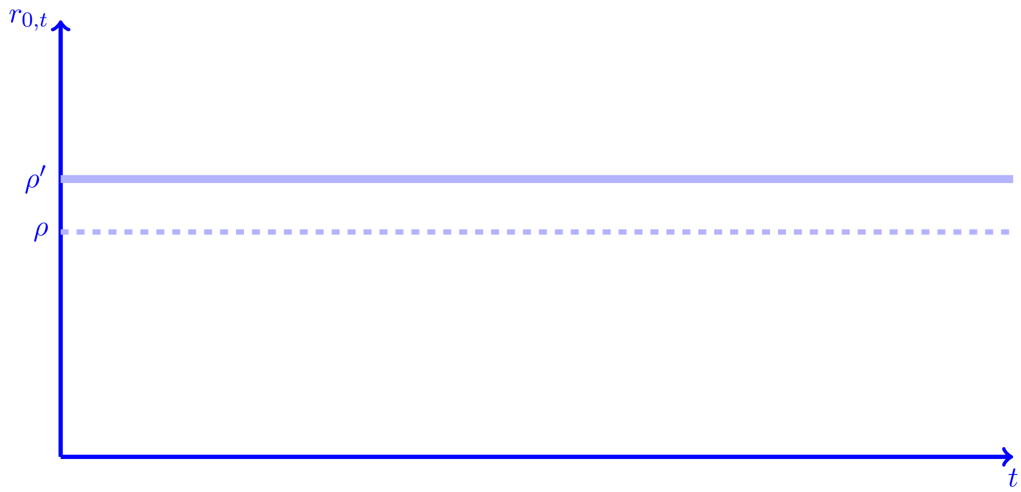
Equilibrium quantities are now $c_t^i = \gamma^t y$, so that

$$q_t^0 = \left(\frac{\beta}{\gamma}\right)^t$$

and therefore

$$r_{0,t} = \frac{\gamma - \beta}{\gamma} \equiv \rho' > \rho.$$

The yield curve is flat again.



5 – Assume now that $y_0^i = y\ \forall i$, $y_t^i = \gamma_1^t y_{t-1}^i$ for $t \leq T$ and $y_t^i = \gamma_2^t y_{t-1}^i$ for $t > T$, with $\gamma_1 > 1$ and $\gamma_2 > 1$. Compute $r_{0,t}\ \forall t$ and plot the yield curve for the two cases $\gamma_1 > \gamma_2$ and $\gamma_1 < \gamma_2$. Discuss.

In this case,

$$y_t = \gamma_1^{\frac{t(t+1)}{2}} y \quad \text{for } t \leq T$$

$$y_t = \gamma_2^{\frac{(t-T))(t-T+1)}{2}} \gamma_2^{\frac{(t-T))T}{2}} \gamma_1^{\frac{T(T+1)}{2}} y \quad \text{for } t > T$$

Therefore

$$q_t^0 = \frac{\beta^t}{\gamma_1^{\frac{t(t+1)}{2}}} \quad \text{for } t \leq T$$

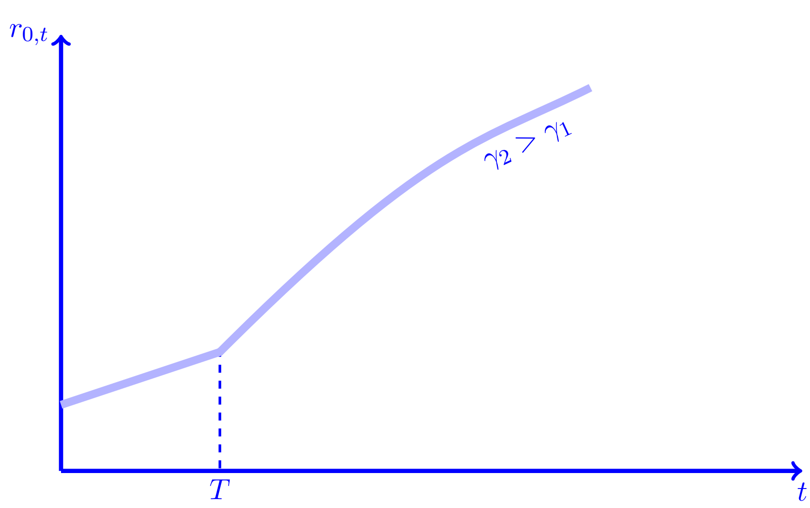
$$q_t^0 = \frac{\beta^t}{\gamma_2^{\frac{(t-T))(t-T+1)}{2}} \gamma_2^{\frac{(t-T))T}{2}} \gamma_1^{\frac{T(T+1)}{2}}} \quad \text{for } t > T$$

and

$$r_{0,t} = -\log\beta + \frac{(t+1)}{2}\log\gamma_1 \quad \text{for } t \leq T$$

$$r_{0,t} = -\log\beta + \underbrace{\frac{T(T+1)}{2t}}_{\text{tends to } 0}\log\gamma_1 \quad \underbrace{-\frac{T(t+1)}{2t}\log\gamma_2}_{\text{negative, tend to } -T/2\log\gamma_2} \quad + \quad \underbrace{\frac{(t+1)}{t}\log\gamma_2}_{\text{dominates in the long run}} \quad \text{for } t > T$$

The path of $r_{0,t}$ is therefore, in the case $\gamma_2 > \gamma_1$:



6 – Let's assume now that there are two agents in the economy, indexed by 1 and 2. They behave in a competitive way. Endowments of agent 1 is $y^1 = \{y+\varepsilon, y-\varepsilon, y+\varepsilon, y-\varepsilon, \ldots\}$ and endowment of agent 2 is $y^1 = \{y-\varepsilon, y+\varepsilon, y-\varepsilon, y+\varepsilon, \ldots\}$, with $|\varepsilon| < y$. Prove that in a competitive equilibrium $\frac{c_t^1}{c_t^2}$ is a constant that depends of $\frac{\mu^1}{\mu^2}$. What does that mean in terms of risk sharing.

FOC wrt c_t^1 and c_t^2:

$$q_t^0 = \frac{\beta^t}{\mu^1 c_t^1},$$

and

$$q_t^0 = \frac{\beta^t}{\mu^2 c_t^2}.$$

Take the ratio to obtain:

$$c_t^2 = \frac{\mu^1}{\mu^2} c_t^1.$$

There is perfect risk sharing.

7 – Compute $\frac{\mu^1}{\mu^2}$ and show that $\frac{\mu^1}{\mu^2} < 1$ when $\varepsilon > 0$. Why?

Take the resource constraint of period t:

$$c_t^1 + c_t^2 = 2y$$

or

$$c_t^1 + \frac{\mu^1}{\mu^2} c_t^1 = 2y$$

which implies

$$c_t^1 = \frac{2}{\left(1 + \frac{\mu^1}{\mu^2}\right)} y$$

Consumption is constant across time. From the FOC of agent 1, we have

$$\frac{q_t^0}{q_0^0} = \beta^t$$

Therefore, the intertemporal budget constraint of agent 1 writes:

$$\sum_{t=0}^{\infty} \beta^t c_t^1 = \sum_{t=0}^{\infty} \beta^t y_t^1$$

or equivalently

$$\frac{2}{\left(1 + \frac{\mu^1}{\mu^2}\right)} y \sum_{t=0}^{\infty} \beta^t = y \sum_{t=0}^{\infty} \beta^t + \left(\varepsilon - \beta\varepsilon + \beta^2\varepsilon + \cdots\right)$$

or

$$\frac{2}{\left(1 + \frac{\mu^1}{\mu^2}\right)} y \frac{1}{1-\beta} = y \frac{1}{1-\beta} + \varepsilon \frac{1}{1+\beta},$$

so that

$$\frac{\mu^1}{\mu^2} = \frac{(1+\beta)y - (1-\beta)\varepsilon}{1+\beta)y + (1-\beta)\varepsilon}$$

We see that $\frac{\mu^1}{\mu^2} > 1$ if $\varepsilon > 0$. Indeed, when $\varepsilon > 0$, agent 1 net present value of endowment is higher than agent 2 one, so that the marginal utility of wealth of agent 1 (μ_1) is lower that the one of agent 2 (μ_2).

IV – Bubbly asset in an Overlapping Generations model

Consider an economy in which overlapping generations of agents live for two periods (young and old). The size of each generation is normalised to one. Preferences are $u(c_t^y) + u(c_{t+1}^o)$. Endowment is ω^y when young and ω^o when old. There exist an asset in quantity A. Each unit of the asset pays dividend $d \geq 0$ every period and has non negative price p_t. In period 0, the old generation holds the A units of the asset. Young agents of period t can save by buying a_{t+1} units of the asset at price p_t or by subscribing a private bond in quantity b_{t+1}. The price of the bond is one, and it pays $(1 + r_{t+1})$ in period $t+1$. The bond is in zero net supply. We assume that the natural debt limit holds. When a young buys one unit of asset, she receives the dividend when old, and then resells the asset.

1 – Write the budget constraints of generation t when young and when old. Use them to prove that in equilibrium, one must have

$$p_t = \frac{p_{t+1} + d}{1 + r_{t+1}}.$$

Budget constraints are

$$\begin{aligned} c_t^y &= \omega^y - p_t a_{t+1} - b_{t+1} \\ c_{t+1}^o &= \omega^o + (p_{t+1} + d)a_{t+1} + (1 + r_{t+1})b_{t+1} \end{aligned}$$

which gives

$$(1 + r_{t+1})c_t^y + c_{t+1}^o = (1 + r_{t+1})\omega^y + \omega^o + \underbrace{\Big(p_{t+1} + d - (1 + r_{t+1})p_t\Big)}_{\mathcal{Q}_{t+1}} a_{t+1}.$$

If $\mathcal{Q}_{t+1} \neq 0$, the household can reach infinite net present value of consumption (the left hand side of the equation) by buying or selling short an infinite amount of asset, which cannot be an equilibrium as supply of goods is finite. One must therefore have in equilibrium the no-arbitrage equation

$$p_t = \frac{p_{t+1} + d}{1 + r_{t+1}}.$$

2 – Derive the Euler equation of a generation t individual

Euler equation is

$$\frac{u'(c_t^y)}{u'(c_{t+1}^o)} = 1 + r_{t+1}$$

3 – Define competitive equilibrium in that economy.

A competitive equilibrium is a sequence of quantities $\{c_t^y, c_t^o, a_{t+1}, b_{t+1}\}$, non negative prices $\{p_t\}$ and interest rate $\{r_{t+1}\}$ such that (i) individual decisions are optimal given prices and (ii) market clears. Optimality of individual behavior is given by the Euler equation and the budget constraints, that collapse into

$$\frac{u'(\omega^y - p_t a_{t+1} - b_{t+1})}{u'(\omega^o + (p_{t+1} + d)a_{t+1} + (1 + r_{t+1})b_{t+1})} = 1 + r_{t+1}.$$

Asset and bond market equilibrium are given by

$$a_{t+1} = A,$$

$$b_{t+1} = 0$$

and the no-arbitrage equation

$$p_t = \frac{p_{t+1} + d}{1 + r_{t+1}}.$$

Ressource constraint implies that aggregate savings of the young are equal to aggregate dissavings of the old, which writes

$$\omega^y - c_t^y = p_t a_t.$$

4 − We now restrict to steady states. Assume that $d > 0$. Show that one cannot have $r \leq 0$ at the steady state. Compute the unique steady state in that case.

Taking the above equations at the steady state implies

$$c^y = \omega_y - pA,$$

$$c^o = \omega^o + (p + d)A$$

and

$$\frac{u'(\omega_t^y - pA)}{u'(\omega^o + (p + d)A)} = 1 + r = \frac{p + d}{p}.$$

$1 + r = \frac{p+d}{p}$ implies $p = \frac{d}{r}$. $r = 0$ cannot be an equilibrium as it would imply $p = +\infty$. $r < 0$ would imply $p < 0$, which cannot be an equilibrium. The steady state is then

$$p = \frac{d}{r},$$

$$c^y = \omega_y - \frac{d}{r}A,$$

$$c^o = \omega^o + \frac{1 + r}{r}dA$$

where r is the solution to

$$\frac{u'(\omega_t^y - \frac{d}{r}A)}{u'(\omega^o + \frac{1+r}{r}dA)} = 1 + r.$$

5 − Assume now that $d = 0$. Show that autarky is always a steady state. Show that when $\omega^y > \omega^o$, there is another steady state in which $p > 0$. Compute that steady state. Why is that a bubbly steady state?

When $d = 0$, the no-arbitrage equation writes

$$p = \frac{p}{1 + r}.$$

$p = 0$ is a solution, in which case the steady state is the autarkic allocation. Another possibility is that $r = 0$. In that case, the Euler equation is

$$\frac{u'(\omega_t^y - pA)}{u'(\omega^o + pA)} = 1,$$

which implies

$$p = \frac{\omega^y - \omega^o}{2A}$$

This second steady state exists only if $\omega^y > \omega^o$, so that $p > 0$. In that case, $c^y = c^o = \frac{\omega^y + \omega^o}{2}$. This is a bubbly equilibrium because the asset is a bubble: it gives no dividends but has a positive price.